On the 3-extendability of quaternary linear codes

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On the 3-extendability of quaternary linear codes

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Abstract

We consider the extendability of linear codes over \( \mathbb{F}_4 \), the field of order four. Let \( C \) be \([n, k, d]_4\) code with \( d \equiv 1 \pmod{4} \), \( k \geq 3 \). The weight spectrum modulo 4 (4-WS) of \( C \) is defined as the ordered 4-tuple \((w_0, w_1, w_2, w_3)\) with

\[ w_0 = \frac{1}{3} \sum_{i > 0} A_i, \quad w_j = \frac{1}{3} \sum_{i \equiv j \pmod{4}} A_i \]

for \( j = 1, 2, 3 \). We prove that \( C \) is 3-extendable if \( w_0 + w_2 = k_2 \) and if either (a) \( w_1 < 4^{k_2} + 4 - \theta_{k_3} \); (b) \( w_1 > 10 \cdot 4^{k_3} - \theta_{k_3} \) or (c) \( (w_0, w_1) = (\theta_{k_3}, 6 \cdot 4^{k_3}) \). We also give a sufficient condition for the l-extendability of \([n, k, d]_4\) codes with \( d \equiv 4 - l \pmod{4} \), \( k \geq 3 \) for \( l = 1, 2, 3 \) when \( w_0 + w_2 = \theta_{k_2} + 2 \cdot 4^{k_2} \).

MSC: 51E20; 94B27

Keywords: linear codes, extension, finite projective spaces, odd sets

1. Introduction

Let \( \mathbb{F}_q \) denote the field of \( q \) elements. We denote by \( \mathbb{F}_q^n \) the set of \( n \)-tuples over \( \mathbb{F}_q \). The weight of a vector \( c \in \mathbb{F}_q^n \), denoted by \( wt(c) \), is the number of nonzero entries in \( c \). An \([n, k, d]_q\) code or a \( q \)-ary linear code of length \( n \) with dimension \( k \) and minimum weight \( d \) is a \( k \)-dimensional subspace of \( \mathbb{F}_q^n \) whose minimum weight of nonzero codewords is \( d \). The weight distribution of \( C \) is the list of numbers \( A_i \) which is the number of codewords of \( C \) with weight \( i \). The weight distribution with \((A_0, A_d, \ldots) = (1, \alpha, \ldots)\) is also expressed as \( 0^d \cdot 1 \cdot \ldots \).

For an \([n, k, d]_q\) code \( C \) with generator matrix \( G \), \( C \) is called \( l \)-extendable if there exist \( l \) vectors \( h_1, \ldots, h_l \in \mathbb{F}_q^k \) such that the extended matrix \([G, h_1^T \cdots h_l^T]\) generates an \([n + l, k, d + l]_q\) code \( C' \), and \( C' \) is an \( l \)-extension of \( C \). Especially when \( l = 1 \), \( C \) is simply called extendable and \( C' \) is an extension of \( C \). In this paper, we deal with the extendability of quaternary linear codes. Extension theorems are employed to find optimal linear codes to construct new codes from...
old ones or to prove the nonexistence of codes with certain parameters; see [8, 15] for ternary linear codes and [2, 11] for linear codes over $\mathbb{F}_q$. The $t$-extendability of $[n,k,d]_4$ codes was investigated in [9, 12] for $t = 1$ when $d$ is odd and in [5, 13, 14, 16] for other cases.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $d \not\equiv 0 \pmod{q}$. We define the weight spectrum modulo $q$ ($q$-WS) as the $q$-tuple $(w_0,w_1,\ldots,w_{q-1})$ with

$$ w_0 = \frac{1}{q} \sum_{i>0} A_i, \quad w_j = \frac{1}{q} \sum_{i \equiv j \pmod{q}} A_i $$

From now on in this section, let $q = 4$. Denote by $\theta_j$ the number of points in $\text{PG}(j,4)$, i.e., $\theta_j = (4^{j+1} - 1)/3$. We set $\theta_0 = 1$ and $\theta_j = 0$ for $j < 0$ for convenience.

As for the known extension theorems for linear codes over $\mathbb{F}_4$, see [5, 14] for the case when $d \equiv 2 \pmod{4}$ and [5, 7, 9, 12] for the case when $d \equiv 3 \pmod{4}$. In this paper, we mainly consider the case when $d \equiv 1 \pmod{4}$. The following result is already known for such a case.

**Theorem 1.1** ([5]). Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $4$-WS $(w_0,\ldots,w_3)$, $k \geq 3$, $d \equiv 1 \pmod{4}$. Then $\mathcal{C}$ is $3$-extendable if one of the following conditions holds:

(a) $w_0 = \theta_{k-4}$,

(b) $w_0 = \theta_{k-3}$ and $w_2 = 3 \cdot 4^{k-2}$,

(c) $w_j = 0$ for $j = 2$ or $3$.

The aim of this paper is to give some new sufficient conditions for the $3$-extendability of $[n,k,d]_4$ codes with $d \equiv 1 \pmod{4}$. We consider the cases $w_0 + w_2 = \theta_{k-2}$ or $\theta_{k-2} + 2 \cdot 4^{k-2}$. The following four theorems are our main results.

**Theorem 1.2.** Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $4$-WS $(w_0,\ldots,w_3)$ with $w_0 + w_2 = \theta_{k-2}$, $k \geq 3$, $d \equiv 1 \pmod{4}$. Then $\mathcal{C}$ is $3$-extendable if either

(a) $w_1 - w_0 < 4^{k-2} + 4 - \theta_{k-3}$ or

(b) $w_1 - w_0 > 10 \cdot 4^{k-3} - \theta_{k-3}$.

Whilst one can not apply Theorem 1.2 when $w_1 - w_0 = 6 \cdot 4^{k-3} - \theta_{k-3}$, we prove the following.

**Theorem 1.3.** Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $d \equiv 1 \pmod{4}$, $k \geq 3$, and $4$-WS $(\theta_{k-3},6 \cdot 4^{k-3},4^{k-2},4^{k-1} - 6 \cdot 4^{k-3})$. Then $\mathcal{C}$ is $3$-extendable.

For two integers $s$ and $t$ with $s > t$, the set of $s$ vectors $v_1,\ldots,v_s \in \mathbb{F}_4^k$ are called $t$-independent if any $t$ of which are linearly independent over $\mathbb{F}_4$.

**Theorem 1.4.** Let $\mathcal{C}$ be an $[n,k,d]_4$ code with $4$-WS $(w_0,\ldots,w_3)$ with $w_0 + w_2 = \theta_{k-2} + 2 \cdot 4^{k-2}$, $k \geq 3$, $d \not\equiv 0 \pmod{4}$ and let $G$ be a generator matrix of $\mathcal{C}$. Then there exist three $2$-independent vectors $a_0,a_1,a_2 \in \mathbb{F}_4^k$ such that the codeword $bG$ has even weight for any vector $b \in \mathbb{F}_4^k$ which is orthogonal to one of $a_0,a_1,a_2$.
Theorem 1.5. Let $C$ and $a_0, a_1, a_2 \in \mathbb{F}_4^k$ be as in Theorem 1.4 and let

$$\mu_i = \frac{1}{3} | \{ b \in \mathbb{F}_4^k \setminus \{(0, \ldots, 0)\} \mid wt(bG) \equiv 2 \pmod{4}, b \perp a_i \} |$$

for $i = 0, 1, 2$. If $\mu_0 + \mu_1 + \mu_2 = w_2$, then $C$ is extendable by adding one of $a_0, a_1, \ldots, a_4$ to $G$ as a column, where $a_4$ and $a_5$ are linearly independent vectors in $\mathbb{F}_4^k$ such that $a_1, a_2, \ldots, a_5$ give a line in $PG(k-1, 4)$. More precisely,

(a) $C$ is 3-extendable by adding $a_0, a_1, a_2$ as columns to $G$ if $d \equiv 1 \pmod{4}$;

(b) $C$ is 2-extendable by adding $a_3$ and $a_4$ as columns to $G$ if $d \equiv 2 \pmod{4}$;

(c) $C$ is extendable by adding one of $a_0, a_1, a_2$ as a column to $G$ if $d \equiv 3 \pmod{4}$.

We give some examples of quaternary linear codes below to which our results can be applied. Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$, where $\omega$ and $\bar{\omega}$ are the roots of $x^2 + x + 1 \in \mathbb{F}_2[x]$. We denote $\omega$ and $\bar{\omega}$ by 2 and 3, respectively, for simplicity.

Example 1.6. Let $C_1$ be the $[14, 3, 9]_4$ code with generator matrix

$$G_1 = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 3 & 0 & 3 & 0 & 2 & 3 & 3 & 0 & 2 & 1 & 1 & 1 & 1
\end{bmatrix}.$$  

Then, $C_1$ has weight distribution $0^1 9^{18} 10^6 11^{24} 12^9 13^6$ with 4-WS (3, 8, 2, 8). $C_1$ is 3-extendable by Theorem 1.2 with condition (a). Actually, we get a $[17, 3, 12]_4$ code with weight distribution $0^4 12^{39} 14^{13} 16^6$ by adding the columns (1, 0, 2)$^T$, (1, 1, 2)$^T$ and (1, 1, 2)$^T$ to $G_1$.

Example 1.7. Let $C_2$ be the $[27, 5, 17]_4$ code with generator matrix

$$G_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 3 & 3 & 0 & 2 & 2 & 3 & 1 & 1 & 1 & 3 & 3 & 0 & 1 & 3 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 3 & 1 & 0 & 0 & 1 & 3 & 3 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 3 & 3 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 3 & 0 & 3 & 1 & 0 & 2 & 2 & 3 & 2 & 0 & 3 & 3 & 0 & 1 & 1
\end{bmatrix}.$$  

Then, $C_2$ has weight distribution

$$0^1 17^{156} 18^{96} 19^{135} 20^{54} 21^{37} 22^{96} 23^{54} 24^{9} 25^{48} 26^{6} 27^3$$

giving 4-WS (21, 192, 64, 64), and is 3-extendable by Theorem 1.2 with condition (b). Adding the columns (1, 1, 2, 0, 3)$^T$, (1, 2, 0, 1, 0)$^T$ and (1, 2, 0, 1, 0)$^T$ to $G_2$, we get an optimal $[30, 5, 20]_4$ code with weight distribution $0^1 20^{435} 24^{531} 28^{57}$.

Example 1.8. Let $C_3$ be the $[22, 4, 9]_4$ code with generator matrix

$$G_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 2 & 0 & 2 & 1 & 3 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 3 & 1 & 3 & 1 & 1 & 2 & 0 & 1 & 1 & 1
\end{bmatrix}.$$  

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Then, $C_3$ has weight distribution
\[0^1 9^3 11^3 13^6 14^2 15^6 16^{15} 17^{63} 18^{36} 19^{51}\]
with 4-WS (5, 24, 16, 40). Hence $C_4$ is 3-extendable by Theorem 1.3. We get a
$[25, 4, 12]_4$ code with weight distribution $0^3 12^1 14^3 16^5 18^2 20^1 22^3 27$ by adding
the columns $(1, 0, 1, 3)^T$, $(0, 1, 1, 1)^T$ and $(0, 1, 1, 1)^T$ to $G_3$.

**Example 1.9.** Let $C_4$ be the $[24, 3, 17]_4$ code with generator matrix
\[
G_4 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
0 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \end{bmatrix}.
\]

Then, $C_3$ has weight distribution $0^1 7^1 8^1 22^1 18^1 19^1 20^1$ giving 4-WS (5, 6, 8, 2), and
one can find $a_0 = (1, 0, 1)$, $a_1 = (1, 0, 2)$, $a_2 = (1, 0, 3)$ as in Theorem 1.4. Since
$\mu_0 = \mu_1 = 2$ and $\mu_2 = 4$, $C_3$ is 3-extendable by Theorem 1.5. Actually, we
construct an optimal $[27, 3, 20]_4$ code with weight distribution $0^3 20^{15} 22^{18}
$ by adding $a_0, a_1, a_2$ to $G_4$ as columns.

**2. Geometric approach**

For an integer $k \geq 3$, let $\Sigma = \text{PG}(k - 1, q)$ be the projective geometry
of dimension $k - 1$ over $\mathbb{F}_q$. A $j$-flat is a projective subspace of dimension $j$ in $\Sigma$.
The 0-flats, 1-flats, 2-flats, 3-flats, $(k - 3)$-flats and $(k - 2)$-flats in $\Sigma$ are called
points, lines, planes, solids, secundum and hyperplanes, respectively. We refer
to [3] and [4] for geometric terminologies. For $j < 0$, a $j$-flat is the empty set
as the usual convention. We investigate linear codes over $\mathbb{F}_q$ through projective
gometry.

Let $C$ be an $[n, k, d]_q$ code with a generator matrix $G$ and let $g_i$ be the $i$-th
row of $G$ ($1 \leq i \leq k$). For $P = (p_1, \ldots, p_k) \in \Sigma$, the weight of $P$ with respect
to $G$, denoted by $w_G(P)$, is defined as
\[w_G(P) = \text{wt}\left(\sum_{i=1}^{k} p_i g_i\right)\].

For a $t$-flat $\Delta$ in $\Sigma$, $w_G(\Delta)$ is defined as $w_G(\Delta) = \sum_{P \in \Delta} w_G(P)$. Let
\[F_d = \{P \in \Sigma \mid w_G(P) = d\}.
\]
Recall that a hyperplane $H$ of $\Sigma$ is defined by a non-zero vector $h = (h_1, \ldots, h_k) \in
\mathbb{F}_q^k$ as $H = \{P(p_1, \ldots, p_k) \in \Sigma \mid h_1 p_1 + \cdots + h_k p_k = 0\}$. $h$ is called the defining
vector of $H$. The following lemma is well-known, see [10, 11].

**Lemma 2.1.** An $[n, k, d]_q$ code $C$ is extendable if and only if there exists a
hyperplane $H$ of $\Sigma$ such that $F_d \cap H = \emptyset$. Moreover, the extended matrix of $G$
by adding the defining vector of $H$ as a column generates an extension of $C$.
Now, let $C$ be an $[n,k,d]_q$ code with $q$-WS $(w_0, w_1, \ldots, w_{q-1})$ and assume $d \not\equiv 0 \pmod{q}$. Let

$$M_i = \{ P \in \Sigma \mid w_G(P) \equiv i \pmod{q} \},$$

$$F = \Sigma \setminus \overline{M_d}.$$ 

Then we have $w_i = |M_i|$ for $0 \leq i \leq q-1$. Note that $F_d \cap M_0 = \emptyset$ and $F_d \subset M_d$. As a corollary of Lemma 2.1, we get the following.

**Corollary 2.2.** $C$ is extendable if there exists a hyperplane $H$ of $\Sigma$ such that $H \subset F$.

We consider the extendability of $C$ from this geometrical point of view. For a line $L = \{P_0, P_1, \ldots, P_q\}$ in $\Sigma$, the weight of $L$ is naturally defined by $w_G(L) = \sum_{i=0}^{q} w_G(P_i)$, which satisfies

$$w_G(L) \equiv 0 \pmod{q},$$

(2.1)

see [16]. The line condition (2.1) determines all possible lines in $\Sigma$.

From now on, we only consider the case when $q = 4$. A $t$-flat $\Pi$ of $\Sigma$ with $|\Pi \cap M_0| = h$, $|\Pi \cap M_1| = i$, $|\Pi \cap M_2| = j$ is called an $(h,i,j)_t$-flat. An $(h,i,j)_1$-flat is called an $(h,i,j)$-line. An $(h,i,j)$-plane, an $(h,i,j)$-solid and so on are defined similarly. Let $A_i$ be the set of all possible $(h,i,j)$ such that an $(h,i,j)_t$-flat exists. Then, it follows from (2.1) that

$$A_1 = \{(0,1,1), (0,3,1), (1,0,0), (1,2,0), (1,4,0), (0,0,3), (0,2,3), (1,1,2), (2,0,1), (2,2,1), (3,1,0), (3,0,2), (1,0,4)\}. \quad (2.2)$$

So, Tables 3.1-3.4 in [12] (the list of all possible $(h,i,j)$-planes with lines for the case $d$ is odd) are also valid for the case $d \equiv 2 \pmod{4}$. Let $M_\epsilon$ be the set of points with even weight, i.e.,

$$M_\epsilon = M_0 \cup M_2.$$ 

If a line $L$ meets $M_\epsilon$ in exactly $i$ points, $L$ is called an $i$-line. From (2.2), there exist only 1-lines, 3-lines or 5-lines of $M_\epsilon$. Such a set in $\Sigma$ is called an odd set or a set of odd type [3]. Thus, the set $M_\epsilon$ forms an odd set in $\Sigma$. Moreover, a 5-line meets $M_0$ in some odd number of points. This yields the following.

**Lemma 2.3.** The set $M_0$ forms an odd set in $M_\epsilon$ if $M_\epsilon$ is a flat in $\Sigma$.

The following three lemmas are needed to prove Theorem 1.2.

**Lemma 2.4** ([14]). An odd set $K$ in $PG(r,4)$ contains a hyperplane of $PG(r,4)$ for $r \geq 2$ if $|K| < \theta_{r-1} + 2 \cdot 4^{r-2}$ or $|K| > \theta_{r-1} + 2 \cdot 4^{r-1} - 4$.

A set $B$ in $PG(r,q)$ is called a blocking set with respect to $s$-flats if every $s$-flat in $PG(r,q)$ meets $B$ in at least one point.
Lemma 2.5 ([1]). Let $\mathcal{B}$ be a blocking set with respect to $s$-flats in $PG(r,q)$. Then, $|\mathcal{B}| \geq \theta_{r-s}$, where the equality holds if and only if $\mathcal{B}$ is an $(r-s)$-flat.

Lemma 2.6 ([5]). Let $C$ be an $[n,k,d]_4$ code with generator matrix $G$, $k \geq 3$, $d \equiv 2$ (mod 4). If $M_0 \cup M_1$ contains a hyperplane $H$ of $\Sigma$, then $C$ is 2-extendable by adding the defining vector of $H$ twice to $G$.

Proof of Theorem 1.2. Let $C$ be an $[n,k,d]_4$ code with generator matrix $G$ with $w_0+w_2 = \theta_{k-2}$, $k \geq 3$, $d \equiv 1$ (mod 4) satisfying $w_1-w_0 < 4^{k-2}+4-\theta_{k-3}$ or $w_1-w_0 > 10 \cdot 4^{k-3}-\theta_{k-3}$. Since $M_e$ is an odd set, $M_e$ is a blocking set with respect to lines. Since $|M_e| = \theta_{k-2}$, $M_e$ forms a hyperplane of $\Sigma$, say $H_1$, by Lemma 2.5. Let $\hat{C}$ be an extended $[n+1,k,d+1]_4$ code with generator matrix $G$, which is given by adding the defining vector of $H_1$ to $G$ and let

$$M_i = \{P \in \Sigma \mid w_G(P) \equiv i \pmod{q}\}.$$

Since $w_G(P) = w_G(P)$ for $P \in H_1$ and $w_G(P) = w_G(P)+1$ for $P \not\in H_1$, we have $\Sigma = M_0 \cup M_2$, $M_0 = M_0 \cup M_3$ and $M_2 = M_1 \cup M_2$. Hence, from Lemma 2.3, $M_0 \cup M_3$ is an odd set. It follows from Lemma 2.4 that $M_0$ contains a hyperplane, say $H_2$, if $w_0+w_3 < \theta_{k-2}+2\cdot 4^{k-3}$ or $w_0+w_3 > \theta_{k-2}+2\cdot 4^{k-2}-4$, which holds from the condition $w_1-w_0 < 4^{k-2}+4-\theta_{k-3}$ or $w_1-w_0 > 10 \cdot 4^{k-3}-\theta_{k-3}$. Hence, $\hat{C}$ is 2-extendable by adding the defining vector of $H_2$ twice to $G$ by Lemma 2.6.

To prove Theorem 1.3, we need the following six lemmas.

Lemma 2.7. Let $\Pi$ be a $(\theta_{t-1}, \varphi_1, 0)_t$ flat in $\Sigma$ for $t \geq 2$. If $\varphi_1 > 0$, then $\varphi_1 \geq 6 \cdot 4^{t-2}$.

Proof. We proceed by induction on $t$. See Table 3.2 in [12] for $t = 2$. Assume $t \geq 3$. Let $\Delta_0$ be $\Pi \cap M_0$. Since $\Delta_0$ is an odd set in $\Pi$, it follows from $|\Delta_0| = \theta_{t-1}$ that $\Delta_0$ is a hyperplane of $\Pi$. Take a $(t-2)$-flat $\delta$ in $\Delta_0$ and let $\Delta_1, \ldots, \Delta_4$ be the other $(t-1)$-flats through $\delta$ in $\Pi$. If all of $\Delta_1, \ldots, \Delta_4$ has a point of $M_1$, it follows from the induction hypothesis that $\varphi_1 \geq 6 \cdot 4^{t-3} \cdot 4 = 6 \cdot 4^{t-2}$. So, we assume that $\Delta_1$ is a $(\theta_{t-2}, 0, 0)_{t-1}$ flat. Take a point $Q \in M_1 \cap \Pi$. We may assume that $Q \in \Delta_2$. Then, every line through $Q$ in $\Pi$ not contained in $\Delta_2$ is a $(1,2,0)$-line. Hence, $\varphi_2 = x + 4^{t-1}$ with $x = |\Delta_2 \cap M_1|$. Take a $(1,2,0)$-line $l$ through $Q$ in $\Pi$ and let $Q'$ be another point of $l \cap M_1$. We may assume that $Q' \in \Delta_3$. Then, we have $\varphi_1 = y + 4^{t-1}$ with $y = |\Delta_3 \cap M_1|$. Hence, $x = y$. If $\Delta_4$ contains no point of $M_1$, then, $x = 4^{t-2}$ and $\varphi_1 = 8 \cdot 4^{t-2}$. Assume that $\Delta_4$ contains a point $Q_1$ of $M_1$. Then, we have $\varphi_1 = z + 4^{t-1}$ with $z = |\Delta_4 \cap M_1|$, whence $x = y = z$. It is easy to see that there is a $(1,2,0)$-line through $Q_1$ in $\Pi$ meeting $\Delta_2$ in a point $R \in M_3$. Counting the number of points in $M_1$ on the lines through $R$, we get $\varphi_1 = x + 2y$. Hence $x = y = z = 2 \cdot 4^{t-2}$ and $\varphi_1 = 6 \cdot 4^{t-2}$.

Lemma 2.8. Let $\Pi$ be a $(\theta_{t-1}, 6 \cdot 4^{t-2}, 0)_t$ flat in $\Sigma$ for $t \geq 2$. For any $(\theta_{t-2}, 0, 0)_{t-2}$ flat $\delta$ in $\Pi$, the $(t-1)$-flats through $\delta$ in $\Pi$ are a $(\theta_{t-1}, 0, 0)_{t-1}$ flat, a $(\theta_{t-2}, 0, 0)_{t-1}$ flat and three $(\theta_{t-2}, 2 \cdot 4^{t-2}, 0)_{t-1}$ flats.
Proof We proceed by induction on $t$. See Table 3.2 in [12] for $t = 2$. Assume $t \geq 3$. Let $\Delta_0$ be $\Pi \cap M_0$. Then, $\Delta_0$ is a $(\theta_{t-1},0,0)_{t-1}$ flat. Take a $(\theta_{t-2},0,0)_{t-2}$ flat $\delta_0$ in $\Delta_0$ and let $\Delta_1, \ldots, \Delta_4$ be the other $(t-1)$-flats through $\delta_0$ in $\Pi$. Suppose that none of $\Delta_1, \ldots, \Delta_4$ is a $(\theta_{t-2},0,0)_{t-1}$ flat in $\Sigma$. Since $\Delta_0$ is a $(\theta_{t-1},0,0)_{t-1}$ flat, all of $\Delta_1, \ldots, \Delta_4$ must be $(\theta_{t-2},6 \cdot 4^{t-3},0)_{t-1}$ flats by Lemma 2.7. From the induction hypothesis, one can take a $(\theta_{t-3},0,0)_{t-2}$ flat $\delta_1$ in $\Delta_1$. Counting the number of points in $M_1$ on the $(t-1)$-flats through $\delta_1$, we get $|\Pi \cap M_1| \geq 6 \cdot 4^{t-3} \cdot \theta_1 > 6 \cdot 4^{t-2}$, a contradiction. Hence, $\Pi$ contains a $(\theta_{t-2},0,0)_{t-1}$ flat, say $H$, and $\delta = \Delta \cap H$ is a $(\theta_{t-2},0,0)_{t-2}$ flat. It follows from the investigation in the proof of Lemma 2.7 that the other $(t-1)$-flats through $\delta$ in $\Pi$ are $(\theta_{t-2},2 \cdot 4^{t-2},0)_{t-1}$ flats.

Lemma 2.9. Let $\Pi$ be a $(\theta_{t-2},4^{t-1},4^{t-1})_t$ flat in $\Sigma$ for $t \geq 2$. Then, $\Pi$ contains a $(\theta_{t-2},0,0)_{t-2}$ flat $\delta$ such that the $(t-1)$-flats through $\delta$ in $\Pi$ are a $(\theta_{t-2},0,4^{t-1})_{t-1}$ flat, a $(\theta_{t-2},4^{t-1},0)_{t-1}$ flat and three $(\theta_{t-2},0,0)_{t-1}$ flats.

Proof Let $\Delta_0$ be $\Pi \cap M_0$. Then, $\Delta_0$ is a hyperplane of $\Pi$, and $\delta = \Delta_0 \cap M_0$ is a $(\theta_{t-2},0,0)_{t-2}$ flat by Lemmas 2.3 and 2.5. Let $\Delta_1, \ldots, \Delta_4$ be the other $(t-1)$-flat through $\delta$ in $\Pi$. We may assume that $\Delta_1$ contains a point $R \in M_3$. Let $\Delta_1$ be a $(\theta_{t-2},x,0)_{t-1}$ flat and let $r_1$ be the number of $(0,i,1)$-lines through $R$ meeting $\Delta_1$ in a point of $M_2$ for $i = 1,3$. Then, we have $r_1 + r_3 = 4^{t-1}$ and $x + r_1 + 3r_3 = 4^{t-1}$, whence $x = r_3 = 0$. Take a $(0,1,1)$-line in $\Pi$ through $R$ meeting $\Delta_2, \Delta_3, \Delta_4$ in $R_2, R_3 \in M_3$ and $Q \in M_1$, respectively. Then, one can deduce that $\Delta_2, \Delta_3$ are $(\theta_{t-2},0,0)_{t-1}$ flats and $\Delta_4$ is a $(\theta_{t-2},4^{t-1},0)_{t-1}$ flat.

Lemma 2.10. Let $\Pi$ be a $(\theta_{t-2},\varphi_1,4^{t-1})_t$ flat in $\Sigma$ for $t \geq 2$. If $\Pi$ contains a $(\theta_{t-2},0,0)_{t-1}$ flat, then $\varphi_1 \in \{4^{t-1},6 \cdot 4^{t-2},3 \cdot 4^{t-1}\}$.

Proof Let $\Delta_0$ be $\Pi \cap M_0$. Then, $\Delta_0$ is a hyperplane of $\Pi$, and $\delta = \Delta_0 \cap M_0$ is a $(\theta_{t-2},0,0)_{t-2}$ flat by Lemmas 2.3 and 2.5. Let $\Delta_1, \ldots, \Delta_4$ be the other $(t-1)$-flat through $\delta$ in $\Pi$. We may assume that $\Delta_1$ is a $(\theta_{t-2},0,0)_{t-1}$ flat. We first assume that $\Delta_2$ is a $(\theta_{t-2},4^{t-1},0)_{t-1}$ flat, i.e., $\Delta_2 \setminus \delta \subset M_1$. If $\Pi \cap M_1 \subset \Delta_2$, then we have $\varphi_1 = 4^{t-1}$. If there is a point $Q \in M_2 \setminus \Delta_2$, say $Q \in \Delta_3$, then the lines in $\Pi$ through $R$ not contained in $\Delta_2$ are $(0,3,1)$-lines, whence $\varphi_1 = 3 \cdot 4^{t-1}$. Finally, assume that all of $\Delta_2, \Delta_3, \Delta_4$ contain a point of $M_2$. Let $\Delta_2$ be a $(\theta_{t-2},x,0)_{t-1}$ flat and let $R$ be a point of $\Delta_2 \cap M_2$. Then, the lines in $\Pi$ through $R$ not contained in $\Delta_2$ are $(0,1,1)$-lines, whence $\varphi_1 = x + 4^{t-1}$. Similarly, each of $\Delta_3, \Delta_4$ contains $x$ points of $M_1$. Hence, we obtain $3x = \varphi_1$, and $\varphi_1 = 6 \cdot 4^{t-2}$.

From the investigation of a $(\theta_{t-2},6 \cdot 4^{t-2},4^{t-1})_t$ flat in the proof of Lemma 2.10, we get the following.

Lemma 2.11. Let $\Pi$ be a $(\theta_{t-2},6 \cdot 4^{t-2},4^{t-1})_t$ flat containing a $(\theta_{t-2},0,0)_{t-1}$ flat $\Delta_1$ in $\Sigma$ for $t \geq 2$. Then, $\Delta_1$ contains a $(\theta_{t-2},0,0)_{t-2}$ flat $\delta$ such that
Lemma 2.13. A \((\theta_{t-2}, 6 \cdot 4^{t-2}, 4^{t-1})_t\) flat \(\Pi\) in \(\Sigma\) contains a \((\theta_{t-2}, 0, 0)_{t-1}\) flat and a \((\theta_{t-2}, 0, 4^{t-1})_{t-1}\) flat through a fixed \((\theta_{t-2}, 0, 0)_{t-2}\) flat for \(t \geq 2\).

Proof. Let \(\Delta_0 = \Pi \cap M_0\) be the \((\theta_{t-2}, 0, 4^{t-1})_{t-1}\) flat. Then, \(\delta_0 = \Delta_0 \cap M_0\) is a \((\theta_{t-2}, 0, 0)_{t-1}\) flat. Let \(\Delta_1, \ldots, \Delta_4\) be the other \((t-1)\)-flats through \(\delta_0\) in \(\Pi\). Suppose that \(\Pi\) contains no \((\theta_{t-2}, 0, 0)_{t-1}\) flat. Since \(|\Pi \cap M_1| = 6 \cdot 4^{t-2}\), we have \(|\Delta_i \cap M_1| = 6 \cdot 4^{t-3}\) for \(i = 1, 2, 3, 4\) by Lemma 2.7. From Lemma 2.8, \(\Delta_1\) contains a \((\theta_{t-3}, 0, 0)_{t-2}\) flat \(\delta_1\). Let \(\pi_1, \ldots, \pi_4\) be the other \((t-1)\)-flats in \(\Pi\) through \(\delta_1\). If \(|\pi_i \cap M_1| = 3 \cdot 4^{t-2}\), then it follows from Lemma 2.10 that

\[|\Pi \cap M_1| = |\Delta_1 \cap M_1| + \sum_{i=1}^4 |\pi_i \cap M_1| \geq 6 \cdot 4^{t-3} + 3 \cdot 4^{t-2} + 3 \cdot 4^{t-2} > 6 \cdot 4^{t-2},\]

a contradiction. Hence, \(|\pi_i \cap M_1| \neq 3 \cdot 4^{t-2}\) for \(i = 1, 2, 3, 4\), and we may assume that \(\pi_1\) is a \((\theta_{t-3}, 6 \cdot 4^{t-3}, 4^{t-2})_{t-1}\) flat and that \(\pi_2, \pi_3, \pi_4\) are \((\theta_{t-3}, 4^{t-2}, 4^{t-2})_{t-1}\) flat. Note that \(F_0 = \delta_1 \cap M_0\) is a \((\theta_{t-3}, 0, 0)_{t-3}\) flat and that the \((t-2)\)-flats in \(\pi_j\) through \(F_0\) are a \((\theta_{t-3}, 0, 4^{t-2})_{t-2}\) flat, a \((\theta_{t-3}, 4^{t-2}, 0)_{t-2}\) flat and three \((\theta_{t-3}, 0, 0)_{t-2}\) flats for \(j = 2, 3, 4\) by Lemma 2.9. Take a \((\theta_{t-3}, 0, 0)_{t-2}\) flat \(\delta_2\) through \(F_0\) in \(\Delta_2\) and a \((\theta_{t-3}, 0, 4^{t-2})_{t-2}\) flat \(\delta_3\) in \(\pi_1\). We may assume that \(\delta_2\) is contained in \(\Delta_2\). Let \(\Delta\) be the \((t-1)\)-flat \(\langle \delta_2, \delta_3 \rangle\) through \(F_0\). Then, the \((t-2)\)-flats in \(\Delta\) through \(F_0\) other than \(\delta_1, \delta_2\) are \((\theta_{t-3}, 2 \cdot 4^{t-3}, 0)_{t-2}\) flats each of which is contained in one of \(\Delta_1, \Delta_3, \Delta_4\). This contradicts that \(\Delta\) meets \(\pi_j\) in \((\theta_{t-3}, 4^{t-2}, 0)_{t-2}\) flat or \((\theta_{t-3}, 0, 0)_{t-2}\) flat for \(j = 2, 3\).

Proof of Theorem 1.3. Let \(C\) be an \([n, k, d]_4\) code with generator matrix \(G\) with 4-WS \((\theta_{k-3}, 6 \cdot 4^{k-3}, 4^{k-2}, 4^{k-1} - 6 \cdot 4^{k-3})\), \(k \geq 3\), \(d \equiv 1 \pmod{4}\). Then, there are a \((\theta_{k-3}, 0, 4^{k-2})_{k-2}\) flat \(H_1\) and a \((\theta_{k-3}, 0, 0)_{k-2}\) flat \(H_2\) through a fixed \((\theta_{k-3}, 0, 0)_{k-3}\) flat \(\delta\) by Lemma 2.12. Let \(\mathcal{C}\) be an extended \([n + 1, k, d + 1]_4\) code with generator matrix \(G\), which is given by adding the defining vector of \(H_1\) to \(G\) and let \(M_t = \{P \in \Sigma \mid w_G(P) \equiv i \pmod{q}\}\). Since \(w_G(P) = w_C(P)\) for \(P \in H_1\) and \(w_G(P) = w_C(P) + 1\) for \(P \not\in H_1, H_2\) is a \((\theta_{k-2}, 0, 0)_{k-2}\) flat for \(\mathcal{C}\). Hence, \(\mathcal{C}\) is 2-extendable by adding the defining vector of \(H_2\) twice to \(G\) by Lemma 2.6.

The following lemma is valid for a \([n, k, d]_4\) code with \(d \equiv 2 \pmod{4}\), which was originally proved for odd \(d\) in [12].

Lemma 2.13. For a plane \(\delta\), \(\delta \cap M_0\) is one of the following:

(a) a line;
(b) a non-singular Hermitian curve \(\mathcal{H}_2\);
(c) the union of three concurrent lines \(\Pi_0 \mathcal{H}_1\);
(d) the plane \(\delta\).
Recall that a line \( l \) of \( \Sigma \) with \( |l \cap M_e| = t \) is called a \( t \)-line. A \( t \)-plane, a \( t \)-solid and so on are defined similarly. The possible planes are 5-, 7-, 9-, 13- and 21-planes by Lemma 2.13.

**Lemma 2.14.** Let \( \Delta \) be a 53-solid, i.e., \( |\Delta \cap M_e| = 53 \). Then, \( \Delta \cap M_e \) consists of three planes through a fixed line.

**Proof.** From Table 2 in [14], \( \Delta \cap M_e \) is either \( \Pi_1 \mathcal{V}_1 \) or \( \mathcal{R}_3 \). Since \( \mathcal{R}_3 \) contains a 11-plane, \( \Delta \cap M_e \) is \( \Pi_1 \mathcal{V}_1 \) by Lemma 2.13.

**Lemma 2.15.** Let \( C \) be an \([n, k, d]_4\) code with \( w_0 + w_2 = \theta_{k-2} + 2 \cdot 4^{k-2} \), \( k \geq 3 \), \( d \not\equiv 0 \pmod{4} \). Then the \((w_0 + w_2)\)-set \( M_e = M_0 \cup M_2 \) consists of three hyperplanes \( H_0, H_1, H_2 \) of \( \Sigma \) through a fixed \((k - 3)\)-flat.

**Proof.** Let \( \Pi \) be a \( t \)-flat in \( \Sigma \) with \( |\Pi \cap M_e| = \theta_{t-1} + 2 \cdot 4^{t-1} \). We shall prove that \( \Pi \cap M_e \) consists of three \((t - 1)\)-flats through a fixed \((t - 2)\)-flat. Our assertion holds for \( t = 2, 3 \) by Lemmas 2.13 and 2.14. Since a line meets \( M_e \) in some odd number of points, one can take a 3-line \( l \) in \( \Pi \). Counting the number of points of \( M_e \) on the planes in \( \Pi \) through \( l \), the equality \((13 - 3)\theta_{t-2} + 3 = \theta_{t-1} + 2 \cdot 4^{t-1} \) implies that all of the planes in \( \Pi \) through \( l \) are 13-planes. Take such two planes \( \delta_1, \delta_2 \) and let \( P_i \) be the vertex of the cone \( \delta_i \cap M_e \), \( i = 1, 2 \). Then, \( \Delta = \langle \delta_1, \delta_2 \rangle \) is a 53-solid with \( \Delta \cap M_e = \Pi_1 \mathcal{V}_1 \). Hence, the line \( \langle P_1, P_2 \rangle \) is contained in \( M_e \).

Let \( S \) be the set of vertices of the cones \( \delta \cap M_e \) for all planes \( \delta \) in \( \Pi \) through \( l \). Then, \( S \) is a \((t - 2)\)-flat. Let \( Q_1, Q_2, Q_3 \) be the points in \( l \cap M_e \). Since \( \langle Q_i, Q \rangle \) is a line contained in \( M_e \) for any \( Q \in S \), \( \langle Q, S \rangle \) is a \((t - 2)\)-flat contained in \( M_e \).

Taking the defining vectors of \( H_0, H_1, H_2 \) as \( a_0, a_1, a_2 \), Theorem 1.4 follows from Lemma 2.15. Furthermore, Theorem 1.5 (a) follows from the following lemma since the condition \( \mu_0 + \mu_1 + \mu_2 = w_2 \) holds if and only if the \((k - 3)\)-flat \( H_0 \cap H_1 \cap H_2 \) is contained in \( M_0 \).

**Lemma 2.16** ([5]). Let \( C \) be an \([n, k, d]_4\) code with generator matrix \( G \), \( k \geq 3 \), \( d \equiv 1 \pmod{4} \). If \( M_e \) contains three distinct hyperplanes \( H_1, H_2, H_3 \) of \( \Sigma \) through a \((k - 3)\)-flat \( \Delta \) with \( \Delta \subset M_0 \), then \( C \) is 3-extendable by adding the defining vectors of \( H_1, H_2, H_3 \) to \( G \).

Theorem 1.5 (c) is obvious. Theorem 1.5 (b) follows from the following.

**Lemma 2.17** ([5]). Let \( C \) be an \([n, k, d]_4\) code with generator matrix \( G \), \( k \geq 3 \), \( d \equiv 2 \pmod{4} \). If \( M_0 \cup M_1 \cup M_3 \) contains two distinct hyperplanes \( H_1, H_2 \) of \( \Sigma \) meeting in a \((k - 3)\)-flat \( \Delta \) with \( \Delta \subset M_0 \cup M_1 \), then \( C \) is 2-extendable by adding the defining vectors of \( H_1, H_2 \) to \( G \).

An alternative method to investigate the \( l \)-extendability of an \([n, k, d]_q\) code \( C \) such that \( A_i > 0 \) implies \( i \equiv 0, -1, \ldots, -s \pmod{q} \), where \( d \equiv -s \) with \( 1 \leq s \leq q - 1 \) is to consider the multiset \( \mathcal{M} = \sum_{i=0}^{s-1} (s - i) \cdot M_i \) consisting of \((s - i)\) copies of \( M_i \) for \( i = 0, 1, \ldots, s - 1 \). In this method, \( C \) is \( l \)-extendable if
and only if \( M \) contains the sum of \( l \) hyperplanes of \( \Sigma \) [6]. The Lemmas 2.6, 2.16, 2.17 can be easily obtained from this approach.

Now, we can look at the extendability of codes in Examples 1.6-1.9 from our geometrical point of view as follows.

**Example 2.18.** Let \( C_1 \) be the \([14,3,9]_4\) code with 4-WS (3,8,2,8) in Example 1.6. Then, \( M_e = M_0 \cup M_2 \) forms a \((3,0,2,2)\)-line whose defining vector is \( (1,0,2) \) in \( \Sigma = \text{PG}(2,4) \). Let \( G_1 \) be the \( 3 \times 15 \) matrix given by adding the column \((1,0,2)^T\) to \( G_1 \). Then, \( G_1 \) generates a \([15,3,10]_4\) code \( \tilde{C}_1 \) with weight distribution \( 0^4 \cdot 10^{24} \cdot 12^{18} \cdot 14^6 \) giving 4-WS \((11,0,10,0)\). The 11-set \( M_0 \) for \( \tilde{C}_1 \) contains a \((5,0,0)\)-line with defining vector \((1,1,2)\). Hence, \( \tilde{C}_1 \) is 2-extendable by adding the column \((1,1,2)^T\) to \( G_1 \) twice.

**Example 2.19.** Let \( C_2 \) be the \([27,5,17]_4\) code with 4-WS (21,192,64,64) in Example 1.7. Then, \( M_e \) forms a hyperplane in \( \Sigma = \text{PG}(4,4) \), whose defining vector is \((1,1,2,0,3)\). Let \( G_2 \) be the \( 5 \times 28 \) matrix given by adding the column \((1,1,2,0,3)^T\) to \( G_2 \). Then, \( G_2 \) generates a \([28,5,18]_4\) code \( \tilde{C}_2 \) with weight distribution \( 0^{18} \cdot 12^{20} \cdot 18^{30} \cdot 24^{62} \cdot 26^{54} \cdot 28^3 \) giving 4-WS \((85,0,256,0)\). The 85-set \( M_0 \) for \( \tilde{C}_2 \) forms a hyperplane with defining vector \((1,2,0,1,0)\). Hence, \( \tilde{C}_2 \) is 2-extendable by adding the column \((1,2,0,1,0)^T\) to \( G_2 \) twice.

**Example 2.20.** Let \( C_3 \) be the \([22,4,9]_4\) code with 4-WS (5,24,16,40) in Example 1.8. Then, \( M_e = M_0 \cup M_2 \) forms a plane, say \( \delta \), whose defining vector is \((1,0,1,3)\) in \( \Sigma = \text{PG}(3,4) \). Let \( G_3 \) be the \( 4 \times 23 \) matrix given by adding the column \((1,0,1,3)^T\) to \( G_3 \). Then, \( G_3 \) generates a \([23,4,10]_4\) code \( \tilde{C}_3 \) with weight distribution \( 0^{10} \cdot 12^{3} \cdot 14^{18} \cdot 16^{4} \cdot 18^{9} \cdot 20^{5} \). Hence, we have \((w_0, w_2) = (45,40)\) for \( \tilde{C}_3 \). The 45-set \( M_0 \) for \( \tilde{C}_3 \) forms an odd set of type \( \Pi_0 \) in \([14]\) containing a plane with defining vector \((0,1,1,1)\). Hence, \( \tilde{C}_3 \) is 2-extendable by adding the column \((0,1,1,1)^T\) to \( G_3 \) twice.

**Example 2.21.** Let \( C_4 \) be the \([24,3,17]_4\) code with 4-WS (5,6,8,2) in Example 1.9. Then, \( M_e \) consists of three lines through a fixed point in \( \Sigma = \text{PG}(2,4) \): \((3,0,2)\)-lines \( \ell_0, \ell_1 \) and a \((1,0,4)\)-line \( \ell_2 \) with defining vectors \( a_0 = (1,0,1), a_1 = (1,0,2), a_2 = (1,0,3) \), respectively. Since \( \mu_0 + \mu_1 + \mu_2 = \sum_{i=0}^{2} |\ell_i \cap M_2| = w_2 \), the point \( \ell_0 \cap \ell_1 \cap \ell_2 \) belongs to \( M_0 \). Hence, \( \tilde{C}_4 \) is 3-extendable by adding \( a_0, a_1, a_2 \) to \( G_4 \) as columns by Lemma 2.16.

**References**


