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Singular value decomposition for a class of linear time-varying systems with application to switched linear systems

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Abstract

The present paper considers singular value decomposition (SVD) for a class of linear time-varying systems. The class considered herein describes time-driven switched linear systems. Based on an appropriate input-output description, the calculation method of singular values and singular vectors is derived. The SVD enables us to characterize the dominant input-output signals using singular vectors, which form orthogonal systems in input and output spaces. The SVD is then applied to switched linear systems to improve the transient response. A numerical example is provided to demonstrate the proposed method.

Keywords: Singular value decomposition, switched linear system, feedforward compensation, linear operator.

1. Introduction

For linear dynamical systems, singular value decomposition (SVD) plays an important role in analysis and control, and SVD has been investigated in a number of studies. For example, model reduction methods for finite-dimensional linear systems have been developed [1, Chapters 7 and 8], and the finite-dimensional approximation problem of a class of infinite-dimensional

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systems has been considered [2]. Exact formulas for singular values and vectors of a system consisting of an SISO inner function and an MIMO rational function have been derived [3]. A compensation signal design method for improving the transient response of linear systems has been derived based on the SVD for linear time-invariant systems [4]. For constrained systems, the compensation signal design problem has been reported [5]. Model predictive control for constrained continuous-time linear systems has also been developed [6]. In these recent studies, singular vectors serve as effective basis vectors for generating approximate continuous-time signals in infinite dimensional function spaces. If the formulas for calculating the singular values and singular vectors for a suitably defined system of interest can be established, SVD-based methods might be applied to various control and approximation problems.

In the present paper, we first consider the SVD of an operator describing the input-output relation of a class of linear time-varying systems and derive a method for calculating singular values and singular vectors. The SVD provides orthogonal input and output sequences that enable us to approximate the original infinite-dimensional input and output spaces using a finite number of singular vectors. The class of systems considered herein represents time-driven switched linear systems, and we consider the compensation signal design problem for switched linear systems with a periodic switching law to improve the transient response using the newly established SVD. The compensation signal design we consider in the present paper is based on a feedforward method and the resulting compensation input over the entire time interval of interest is computed off-line using the desired and uncompensated responses.

The remainder of the present paper is organized as follows. In Section 2, the SVD of an operator describing a class of linear time-varying systems is considered, and the calculation method of singular values and singular vectors is derived. In Section 3, the obtained SVD is applied to a switched linear system with a periodic switching law, and the compensation law for improving the transient response is considered. A numerical example is presented in Section 4 to illustrate the fundamental properties of the proposed method.

2. Singular Value Decomposition for a Class of Linear Time-varying Systems

Consider a class of linear time-varying systems defined over a finite horizon $[0, h]$:

$$\Sigma : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)v(t), & x(0) = 0 \\ z(t) = E(t)x(t) \end{cases} \quad (1)$$

$$(A(t), B(t), E(t)) := \begin{cases} (A_1, B_1, E_1) & 0 \leq t < t_1 \\ (A_2, B_2, E_2) & t_1 \leq t < t_2 \\ \vdots & \vdots \\ (A_N, B_N, E_N) & t_{N-1} \leq t \leq t_N \\ t_0 = 0 < t_1 < t_2 < \dots < t_N = h \end{cases}$$

where $x(t) \in \mathbb{R}^{n_x}$, $v(t) \in \mathbb{R}^{n_v}$, and $z(t) \in \mathbb{R}^{n_z}$ denote the state, input, and output, respectively. This system represents a time-dependent switched linear system. Switched linear systems with a periodic switching law can be described by this particular form (additional details are presented in Section 3). In this section, starting with the introduction of appropriate generalized input and output spaces, we derive the singular value decomposition method for the system given in (1), which will be used for the transient improvement of switched linear systems.

First, define Hilbert spaces $\mathcal{V} := L_2(0, h; \mathbb{R}^{n_v})$ and $\mathcal{Z} := \mathbb{R}^{n_x} \times L_2(0, h; \mathbb{R}^{n_z})$ with the inner products

$$\langle f_1, f_2 \rangle_{\mathcal{V}} := \int_0^h f_1^T(\beta) f_2(\beta) d\beta, \quad f_1, f_2 \in \mathcal{V}, \quad (2)$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} := g_1^{0T} g_2^0 + \int_0^h g_1^{1T}(\beta) g_2^1(\beta) d\beta,$$

$$g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \quad (3)$$

and denote the input and output in \mathcal{V} and \mathcal{Z} as

$$\begin{aligned} v &\in \mathcal{V}, \quad \hat{z} := \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{Z}, \\ z^0 &:= Fx(h), \quad F \in \mathbb{R}^{n_x \times n_x}, \\ z^1(t) &:= z(t), \quad 0 \leq t \leq h \end{aligned} \quad (4)$$

where F is a weighting matrix for the terminal state. The relationship between v and \hat{z} is given by a linear operator $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$:

$$\begin{aligned}\hat{z} &= \Gamma v \\ \Gamma v &:= \begin{bmatrix} (\Gamma v)^0 \\ (\Gamma v)^1 \end{bmatrix},\end{aligned}$$

$$\begin{aligned}(\Gamma v)^0 &= F \sum_{s=1}^{N-1} \Phi_{(N,s)}(h) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau \\ &\quad + F \int_{t_{N-1}}^h e^{A_N(h-\tau)} B_N v(\tau) d\tau,\end{aligned}\tag{5}$$

$$\begin{aligned}(\Gamma v)_{[t_k, t_{k+1}]}^1(t) &= E_{k+1} \sum_{s=1}^k \Phi_{(k+1,s)}(t) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau \\ &\quad + E_{k+1} \int_{t_k}^t e^{A_{k+1}(t-\tau)} B_{k+1} v(\tau) d\tau,\end{aligned}\tag{6}$$

$$t_k \leq t \leq t_{k+1}, \quad k = 0, \dots, N-1,$$

$$\Phi_{(\ell,m)}(t) := e^{A_\ell(t-t_{\ell-1})} e^{A_{\ell-1}(t_{\ell-1}-t_{\ell-2})} \dots e^{A_{m+1}(t_{m+1}-t_m)},$$

$$\ell, m \in \mathbb{Z}_+ : \ell > m \geq 0,$$

$$t_{\ell-1} \leq t \leq t_\ell.$$

Note that $(\Gamma v)^0$ and $(\Gamma v)^1$ denote the weighted terminal state and output signal corresponding to the input signal v over $[0, h]$, respectively. For the operator Γ , we consider the following singular value problem:

$$\begin{aligned}\Gamma f &= \sigma g, \quad \Gamma^* g = \sigma f, \\ \sigma &\in \mathbb{R}, f \in \mathcal{V}, g \in \mathcal{Z}, (f \neq 0, g \neq 0).\end{aligned}\tag{7}$$

The singular vectors f and g represent the input and output signals in \mathcal{V} and \mathcal{Z} . The pairs (f, g) corresponding to the larger singular values σ characterize the dominant input–output behavior of the system Σ . The following theorem provides a calculation method of the singular values σ and the explicit characterization of the singular vectors f and g that satisfy the relation (7).

Theorem 1. *The singular values are given by the roots of the following*

transcendental equation:

$$\det \{M(\sigma)\} = 0, \quad (8)$$

$$M(\sigma) := \left[-\frac{1}{\sigma} F^T F \quad I \right] e^{J_N(\sigma)\bar{d}_N} e^{J_{N-1}(\sigma)\bar{d}_{N-1}} \dots e^{J_1(\sigma)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$J_m(\sigma) := \begin{bmatrix} A_m & \frac{1}{\sigma} B_m B_m^T \\ -\frac{1}{\sigma} E_m^T E_m & -A_m^T \end{bmatrix}, \bar{d}_m := t_m - t_{m-1}, \quad m = 1, 2, \dots, N.$$

Let σ_i be a singular value. Then, the corresponding singular vectors $f_i \in \mathcal{V}$ and $g_i \in \mathcal{Z}$ are given as follows:

$$f_{i[t_{k-1}, t_k]}(\beta) = \frac{1}{\sigma_i} \begin{bmatrix} 0 & B_k^T \end{bmatrix} e^{J_k(\sigma_i)(\beta - t_{k-1})} e^{J_{k-1}(\sigma_i)\bar{d}_{k-1}} \dots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \quad k = 1, 2, \dots, N, \quad (9)$$

$$g_i^0 = \frac{1}{\sigma_i} \begin{bmatrix} F & 0 \end{bmatrix} e^{J_N(\sigma_i)\bar{d}_N} e^{J_{N-1}(\sigma_i)\bar{d}_{N-1}} \dots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \quad (10)$$

$$g_{i[t_{k-1}, t_k]}^1(\beta) = \frac{1}{\sigma_i} \begin{bmatrix} E_k & 0 \end{bmatrix} e^{J_k(\sigma_i)(\beta - t_{k-1})} e^{J_{k-1}(\sigma_i)\bar{d}_{k-1}} \dots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \quad k = 1, 2, \dots, N, \quad (11)$$

$$q_i \neq 0 : M(\sigma_i)q_i = 0.$$

Proof. For the operator Γ , the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ is calculated as follows (see the Appendix for details):

$$(\Gamma^* \hat{z})_{[t_{k-1}, t_k]}(\tau) = \begin{cases} B_k^T e^{A_k^T(t_k - \tau)} \Phi_{(N,k)}^T(h) F^T \hat{z}^0 \\ + \sum_{s=k}^{N-1} \int_{t_s}^{t_{s+1}} B_k^T e^{A_k^T(t_k - \tau)} \Phi_{(s+1,k)}^T(\beta) E_k^T \hat{z}^1(\beta) d\beta \\ + \int_{\tau}^{t_k} B_k^T e^{A_k^T(\beta - \tau)} E_k^T \hat{z}^1(\beta) d\beta \\ t_{k-1} \leq \tau \leq t_k, \quad k = 1, 2, \dots, N-1, \\ B_N^T e^{A_N^T(h - \tau)} F^T \hat{z}^0 + \int_{\tau}^{t_N} B_N^T e^{A_N^T(\beta - \tau)} E_N^T \hat{z}^1(\beta) d\beta \\ t_{N-1} \leq \tau \leq t_N, \quad k = N \end{cases} \quad (12)$$

$$\hat{z} = \begin{bmatrix} \hat{z}^0 \\ \hat{z}^1 \end{bmatrix} \in \mathcal{Z}.$$

By introducing the auxiliary variables

$$\begin{aligned}
p_1(t) &:= \int_0^t e^{A_1(t-\tau)} B_1 v(\tau) d\tau, \quad 0 \leq t \leq t_1, \\
p_k(t) &:= e^{A_k(t-t_{k-1})} p_{k-1}(t_{k-1}) + \int_{t_{k-1}}^t e^{A_k(t-\tau)} B_k v(\tau) d\tau, \\
&\quad t_{k-1} \leq t \leq t_k, \quad k = 2, 3, \dots, N, \\
q_N(t) &:= e^{A_N^\top(t_N-t)} F^\top \hat{z}^0 + \int_t^{t_N} e^{A_N^\top(\beta-t)} E_N^\top \hat{z}^1(\beta) d\beta, \\
q_k(t) &:= e^{A_k^\top(t_k-t)} q_{k+1}(t_k) + \int_t^{t_k} e^{A_k^\top(\beta-t)} E_k^\top \hat{z}^1(\beta) d\beta, \\
&\quad t_{k-1} \leq t \leq t_k, \quad k = 1, 2, \dots, N-1
\end{aligned}$$

to (5), (6), and (12), the relation (7) is rewritten as the following set of equations:

$$\dot{p}_k(t) = A_k p_k(t) + B_k v(t), \quad (13)$$

$$\sigma \hat{z}^1(t) = E_k p_k(t), \quad (14)$$

$$t_{k-1} \leq t \leq t_k, \quad k = 1, 2, \dots, N,$$

$$\sigma \hat{z}^0 = F p_N(t_N), \quad (15)$$

$$p_1(0) = 0, \quad (16)$$

$$\dot{q}_k(t) = -A_k^\top q_k(t) - E_k^\top \hat{z}^1(t), \quad (17)$$

$$t_{k-1} \leq t \leq t_k, \quad k = 1, 2, \dots, N,$$

$$\sigma v(t) = B_k^\top q_k(t), \quad (18)$$

$$q_N(t_N) = F^\top \hat{z}^0. \quad (19)$$

By eliminating v and \hat{z}^1 from the differential equations (13) and (17) using (14) and (18), (13) and (17) yield the following differential equation:

$$\begin{bmatrix} \dot{p}_k(t) \\ \dot{q}_k(t) \end{bmatrix} = \begin{bmatrix} A_k & \frac{1}{\sigma} B_k B_k^\top \\ -\frac{1}{\sigma} E_k^\top E_k & -A_k^\top \end{bmatrix} \begin{bmatrix} p_k(t) \\ q_k(t) \end{bmatrix}.$$

The solution to this differential equation on $[t_{k-1}, t_k]$ is given by

$$\begin{bmatrix} p_k(t_k) \\ q_k(t_k) \end{bmatrix} = e^{J_k(\sigma)(t_k-t_{k-1})} \begin{bmatrix} p_{k-1}(t_{k-1}) \\ q_{k-1}(t_{k-1}) \end{bmatrix}.$$

From equations (15) and (19), the boundary condition $q_N(t_N) = \frac{1}{\sigma} F^T F p_N(t_N)$, which implies

$$\left[-\frac{1}{\sigma} F^T F, I\right] \begin{bmatrix} p_N(h) \\ q_N(h) \end{bmatrix} = 0,$$

is obtained. Then, we have

$$\begin{aligned} & \left[-\frac{1}{\sigma} F^T F \quad I\right] \begin{bmatrix} p_N(h) \\ q_N(h) \end{bmatrix} \\ &= \left[-\frac{1}{\sigma} F^T F \quad I\right] e^{J_N(\sigma)\bar{d}_N} \begin{bmatrix} p_{N-1}(t_{N-1}) \\ q_{N-1}(t_{N-1}) \end{bmatrix} \\ &= \dots \\ &= \left[-\frac{1}{\sigma} F^T F \quad I\right] e^{J_N(\sigma)\bar{d}_N} e^{J_{N-1}(\sigma)\bar{d}_{N-1}} \dots e^{J_1(\sigma)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_1(0) \\ &= M(\sigma)q_1(0) \\ &= 0. \end{aligned} \tag{20}$$

Since it has been shown that $q_1(0) \neq 0$ iff $f \neq 0$, and $g \neq 0$, the matrix $M(\sigma)$ in (20) must be singular for the singular value σ . Therefore, the singular values are given by the roots of equation (8). The singular vectors (9) through (11) corresponding to the singular value σ are constructed by expressing v , z^0 , and z^1 using the auxiliary variables p_k and q_k . \square

Remark 2. *The singular vectors $\{f_i\}$, and $\{g_i\}$ form orthogonal sequences in spaces \mathcal{V} and \mathcal{Z} . The singular vectors f_i corresponding to the large singular values describe the input signals over $[0, h]$ that have a significant effect on the input-output dynamics of the linear time-varying system (1) because the output \hat{z} is given as $\sigma_i \cdot g_i$, which indicates that the output g_i is multiplied by σ_i when f_i is applied to the system.*

Remark 3. *The singular values are calculated by using general methods such as the bisection algorithm. Once the singular values are calculated from the determinant equation (8), the singular vectors are easily computed from (9) through (11) because they have the form of the autonomous response of a switched linear system. Although there are infinitely many discrete singular values, which approaches zero, we use only larger singular values and the*

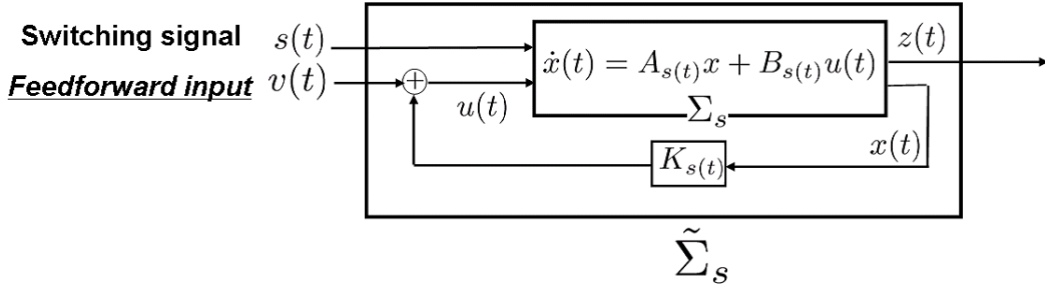


Figure 1: Switched linear system with compensation.

corresponding singular vectors in the compensation design. When a singular value σ_i is found from (8), it may be necessary to check whether σ_i is the largest root. In [7], the method for counting the zeros (roots) for a similar type of equation on a given interval has been considered based on a winding number method in complex analysis. This approach is also applicable to the present case.

Based on the SVD for switched linear systems obtained in this section, we derive a feedforward compensation signal design method for switched linear systems.

In the following, we normalize the singular vectors as $\|f_i\|_{\mathcal{V}} = 1$, $\|g_i\|_{\mathcal{Z}} = 1$ and denote the singular values by $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots$ in decreasing order and the corresponding singular vectors by f_i and g_i ($i = 1, 2, \dots$).

3. Transient Improvement for a Switched Linear System with Periodic Switching

A switched system consists of several subsystems and a switching signal that determines the transition of the dynamics among the subsystems. The stabilizing switching law, system structures (controllability and observability), and optimal control have been investigated in a number of studies [8, 9, 10, 11, 12, 13].

From practical viewpoint, there are many systems which are formulated as switched systems. In addition, some control problems can be formulated in the framework of switched systems, e.g., multi-controller scheme for systems with large uncertainty (see, for example, [8, 10, 14]). A typical example of a

switched system is an electronic circuit with an on-off switch realized using transistor (e.g., DC-DC converter). In this system, the on-off switch acts as a discrete switching signal, which corresponds to switch the subsystem, and a voltage (or current) input signal corresponds to a continuous-time input signal. A simple model of a manual transmission is also an example of switched systems, where the gear shift position is a discrete-valued input and the input acceleration is a continuous-time signal [15]. Two distinctive switching laws for switched systems are the time-driven switching law and the even-driven (state-dependent) switching law, and various switching laws have been proposed so far [8]. When the event-driven switching law is applied to a switched system, the resulting system is considered as a hybrid system whose dynamics changes depending on the value of the state. In the case of the time-dependent switching law, the resultant system is considered as a time-varying system. In the control of these systems, the relationship between the switching of subsystems and continuous-time signal has to be taken into account to design control inputs. The SVD considered in the previous section provides a way to clarify this relationship for, in particular, switched linear systems with the time-driven switching law.

In the optimal control of the switched systems, the optimal design of both the switching signal and continuous-time input is by nature a difficult problem [8]. Moreover, if one pursues optimality exclusively, the switching frequency might become very high or the switching law might require infinitely many switchings on a finite interval [12, 13], which is not desirable from a practical viewpoint. Instead of designing the switching signal to achieve good performance, we consider the feedforward compensation signal design problem.

In this section, based on the SVD for the class of linear time-varying systems developed in the previous section, we derive a method for improving the transient response of a switched linear system with a periodic switching law. The compensation signal is computed off-line beforehand based on uncompensated and desired transient response over the finite time interval of interest.

First, we consider the following switched linear system:

$$\Sigma_s : \begin{cases} \dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t) \\ z(t) = E_{s(t)}x(t) \end{cases} \quad (21)$$

where $x(t)$, $u(t)$, and $z(t)$ denote the state, input, and output, respectively. Subscript $s(t) \in \mathcal{S} := \{1, 2, \dots, r\}$ denotes the switching signal, where r is

the number of the subsystems. When we apply

$$u(t) = K_{s(t)}x(t) + v(t) \quad (22)$$

to Σ_s , the resulting system $\tilde{\Sigma}_s$ (Fig. 1) is described by

$$\tilde{\Sigma}_s : \begin{cases} \dot{x}(t) = \tilde{A}_{s(t)}x(t) + B_{s(t)}v(t) \\ z(t) = E_{s(t)}x(t) \\ \tilde{A}_{s(t)} := A_{s(t)} + B_{s(t)}K_{s(t)}. \end{cases}$$

In (22), $v(t)$ denotes the compensation signal that we will design to improve the transient response. If feedback gain matrices K_i and coefficients ω_i exist such that the following convex combination of \tilde{A}_i is Hurwitz:

$$A_0 = \sum_{i \in \mathcal{S}} w_i \tilde{A}_i \quad (\omega_i \geq 0, \sum_{i \in \mathcal{S}} \omega_i = 1), \quad (23)$$

then the system can be stabilized via periodic switching signal as follows [8, 9]. Let K_i and w_i be one pair that satisfies (23). Then, for a small $T > 0$, the periodic switching signal

$$s(t) = \begin{cases} 1, & t \in [kT, (k + \omega_1)T) \\ 2, & t \in [(k + \omega_1)T, (k + \omega_1 + \omega_2)T) \\ \vdots & \vdots \\ r, & t \in [(k + 1 - \omega_r)T, (k + 1)T) \end{cases} \quad k = 0, 1, \dots$$

stabilizes the system exponentially because matrix $A_d := e^{\omega_r A_r T} \dots e^{\omega_1 A_1 T} = e^{A_0 T + f(T)T^2}$ (f is an analytic and bounded matrix-valued function) can be stable for $T > 0$, and $x((k + 1)T) = A_d x(kT)$ holds. Consequently, a system with the periodic switching signal can be represented in the form of the linear time-varying system (1). Note that the switching interval of the above switching law is given by $\omega_i \cdot T$. If the lower limit $T_{low} > 0$ for the switching interval is imposed, one needs to pay attention to the choice of ω_i and T so that the condition $\omega_i \cdot T \geq T_{low}$ is satisfied. From the implementation viewpoint, the switching control law is relatively simple because of its periodicity, and it is especially applicable to the systems where the switching rule cannot be arbitrarily modified during operation due to limitations on hardware and/or software.

For the system stabilized by the periodic switching signal, we consider the transient improvement by the compensation input $v(t)$ over the finite horizon $0 \leq t \leq h$ in the sense that the resulting system response approaches a certain prescribed response. Let

$$\hat{y}_d = \begin{bmatrix} x_d(h) \\ y_{d[0,h]} \end{bmatrix} \in \mathcal{Z}$$

be a desired output in \mathcal{Z} (pair of the terminal state and the output signal), and let

$$\hat{y} = \begin{bmatrix} x(h) \\ y_{[0,h]} \end{bmatrix} \in \mathcal{Z}$$

be the nominal response (without compensation input v). The desired response \hat{y}_d can be chosen from the response of a certain reference model that has a good transient property. Define the error $\hat{e} := \hat{y}_d - \hat{y} \in \mathcal{Z}$. We design compensation input $v(t)$ such that the resulting system response closely approximates the desired response \hat{y}_d . We construct the compensation input $v(t)$ by the linear combination of the singular vectors $f_i \in \mathcal{V}$ corresponding to $N_s > 0$ singular values in decreasing order, which represent the dominant input signals over $[0, h]$ in \mathcal{V} . Since N_s singular vectors g_1, g_2, \dots, g_{N_s} form the orthonormal system in \mathcal{Z} , the closest element of $\text{span}\{g_1, g_2, \dots, g_{N_s}\}$ to \hat{e} is given by

$$\hat{e}' := \sum_{i=1}^{N_s} \langle \hat{e}, g_i \rangle_{\mathcal{Z}} \cdot g_i. \quad (24)$$

Consequently, by applying the compensation input

$$v_{[0,h]} = \sum_{i=1}^{N_s} \frac{1}{\sigma_i} \langle \hat{e}, g_i \rangle_{\mathcal{Z}} \cdot f_i, \quad (25)$$

the error is minimized in the combination of N_s singular vectors f_i . When the compensation input (25) is applied, the approximation error $\epsilon := \|\hat{e} - \hat{e}'\|_{\mathcal{Z}}$ is given by

$$\begin{aligned} \epsilon^2 &= \|\hat{e} - \sum_{i=1}^{N_s} \langle \hat{e}, g_i \rangle_{\mathcal{Z}} \cdot g_i\|_{\mathcal{Z}}^2 \\ &= \|\hat{e}\|_{\mathcal{Z}}^2 - \sum_{i=1}^{N_s} \|\langle \hat{e}, g_i \rangle_{\mathcal{Z}}\|_{\mathcal{Z}}^2. \end{aligned}$$

Thus, ϵ decreases as the number N_s of singular values and vectors used increases. The number N_s is a design parameter for approximating the error $\hat{\epsilon}$. Note that although the error decreases monotonically as N_s increases, the compensation signal might become larger in order to eliminate only small deviation. There is no single definite way to determine the dimension (the number of singular values and vectors). However, as the error decreases as N_s increases, we can possibly use the decrease rate in the error ϵ as an index for choosing N_s . Let ϵ_{N_s} be the error when N_s singular values and vectors are employed, and choose $\bar{\epsilon} > 0$ for a threshold of the decrease rate of the error. Then, by increasing N_s , employ the smallest number N_s that satisfies the inequality $\epsilon_{N_s} - \epsilon_{N_s+1} < \bar{\epsilon}$.

Remark 4. *Note that “transient improvement” is used in the sense that the output approaches the required output. Thus, the improvement implies a small error between these two outputs. The choice of the desired output depends on the control problem under consideration. The desired output can be chosen from any time-parameterized signals, which could be artificially constructed or generated from the output of a reference model having desired properties (e.g., small undershoot for a specific control variable, settling time). The method is useful especially when the output signal needs to be shaped in the time-domain. We can also deal with the problem in which the output is required to pass prescribed points over the horizon.*

Remark 5. *The other bases might be employed in the algorithm to give a compensation input; however, the singular vectors have some features that the other base vectors do not have. The singular vectors are derived by solving the singular value problem of the linear time-varying system under consideration. Therefore, they have a direct relationship with the system for which the compensation input is designed; the singular values provide an index of influence on the input-output relationship of the system for the corresponding singular vectors (note that the norm of the output \hat{z} is $\|\hat{z}\|_{\mathcal{Z}} = \|\Gamma f_i\|_{\mathcal{Z}} = \|\sigma_i g_i\|_{\mathcal{Z}} = \sigma_i$ for the normalized singular vectors $f_i \in \mathcal{V}, g_i \in \mathcal{Z}$, see also Remark 2). Therefore, by employing the singular vectors corresponding to the singular values in a decreasing order, the subspace spanned by the singular vectors consists of the dominant signals for the system. This would result in a relatively small subspace dimension to construct the compensation input as compared to that in the case when the other bases are employed. In the transient improvement problem here, the required signal is first approximated by the linear combination of the singular vectors $g_i \in \mathcal{Z}$. Then, the compensation input v*

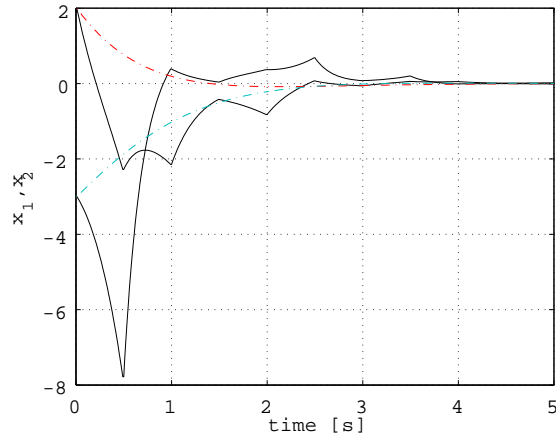


Figure 2: State response (solid) and response of the average system (dashed).

generating such required signal can be simply given by the linear combination of the corresponding singular vectors, $f_i \in \mathcal{V}$. Thus, the design procedure is fairly simplified by employing the singular vectors f_i, g_i as bases in input and output spaces. The compensation signal mentioned previously might become larger as N_s increases when not only the singular vectors but also the other base vectors are employed. This is mainly attributed to the relationship between the properties of the dynamical system and the required signal and not the choice of basis employed.

Remark 6. For standard linear time-invariant systems defined on finite interval, the SVD-based compensation method using the orthogonal expansion technique has been considered in [4]. We have derived herein a compensation method for switched linear systems using the newly derived SVD for switched systems.

4. Numerical Example

We next present a simple numerical example to illustrate the fundamental properties of the proposed SVD-based compensation method.

Consider the following feedback-compensated switched linear system:

$$\begin{aligned} \tilde{\Sigma}_s \begin{cases} \dot{x}(t) = \tilde{A}_{s(t)}x(t) + B_{s(t)}v(t) \\ z(t) = E_{s(t)}x(t) \end{cases} & \quad (26) \\ \mathcal{S} = \{1, 2\}, \tilde{A}_1 = \begin{bmatrix} -3 & 2 \\ 1 & 2 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix} \\ B_1 = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Both subsystems are unstable and have the eigenvalues $(-3.37, 2.37)$ and $(-5.46, 1.46)$. We first design the stabilizing periodic switching law according to the method described in Section 3. When we choose the coefficients $w_1 = 0.5$ and $w_2 = 0.5$ in (23), the matrix $A_0 = w_1\tilde{A}_1 + w_2\tilde{A}_2$ becomes Hurwitz and can therefore be stabilized by periodic switching. The stable matrix $A_d = e^{0.5 \cdot A_2 \cdot 1} \cdot e^{0.5 \cdot A_1 \cdot 1}$ (eigenvalues: $0.135 \pm 0.2527i$) is obtained for $T = 1$, and the periodic switching law

$$s(t) = \begin{cases} 1, & t \in [kT, (k+0.5) \cdot T) \\ 2, & t \in [(k+0.5) \cdot T, (k+1) \cdot T) \end{cases} \\ k = 0, 1, 2, \dots \quad (27)$$

exponentially stabilizes the system. Figure 2 shows the state response (x_1, x_2) for the initial condition $x(0) = [2, -3]^T$ with the periodic switching law (solid line). The response of the average system $(\dot{x}(t) = A_0x(t))$ is also shown as a dashed line. Although the response converges to zero, the behavior is not smooth at switching instants and sharp changes can be observed. Next, we design compensation signal v , which improves this response in the sense that the resulting response of the system comes to resemble that of the average system. Although we use the response of the average system here, any response can be used. Note that the response comes to resemble that of the average system as T in (27) decreases. However, in such a case, much more frequent switching between subsystems A_1 and A_2 is required.

To design the compensation signal, we first compute the singular values and singular vectors. For parameters $h = 4$ (compensation period) and $N = 8$, the singular values are computed from Theorem 1 (Fig. 3). The matrix F in (4) is chosen by the solution to the Lyapunov equation $F^T F = P : A_0^T P + P A_0 + I = 0$. Figures 4 and 5 depict the normalized singular vectors $f_i \in \mathcal{V}$ and $g_i^1 \in L_2(0, h; \mathbb{R}^2)$ ($i = 1, 2, 3, 9, 10$). The singular

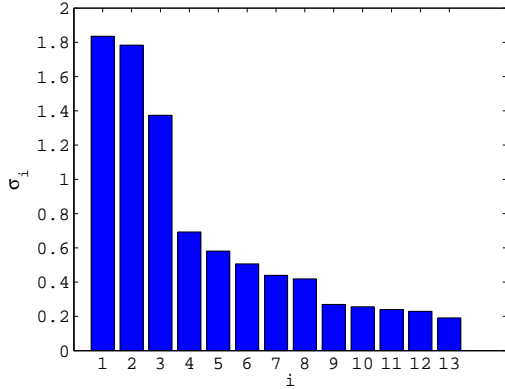


Figure 3: Singular values $\sigma_1, \dots, \sigma_{13}$.

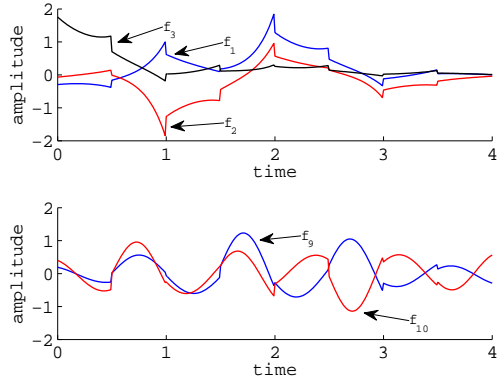


Figure 4: Normalized singular vectors $f_i \in L_2(0, h; \mathbb{R})$.

vectors corresponding to the small singular values σ_9 , and σ_{10} tend to exhibit oscillating behaviors. As mentioned in Remark 2, when the singular vector f_1 corresponding to the largest singular value is applied to the system (26) ($v(t) = f_1(t)$), the output response $z = x$ is given by $\sigma_1 \cdot g_1^1(t) = 1.8353 \cdot g_1^1(t)$, which implies that the system generates a magnified signal of $g_1^1(t)$. Note also that the singular value σ_i represents the value of the norm of the output g_i corresponding to the unit energy input $v = f_i$. When the normalized singular vector f_{10} corresponding to small singular value σ_{10} is applied to the system, $\sigma_{10} \cdot g_{10}^1(t) = 0.2563 \cdot g_{10}^1(t)$ is generated in the output, which implies that the the singular vectors corresponding to the smaller singular values have less of an effect on the input–output relation. Using seven ($N_s = 7$) singular values $\sigma_1, \sigma_2, \dots, \sigma_7$, we design the compensation signal v according to the method considered in Section 3. Figure 6 shows the compensation signal. The amplitude around the initial time is larger to improve the large deviation of the original response. Figure 7 shows the state response with compensation. The response approaches that of the average system.

The responses for $N_s = 13$ are also shown in Figs. 6 and 7. Although the compensation signal for $N_s = 13$ exhibits similar behavior near the initial time, the amplitude is larger than that of the compensation signal for $N_s = 7$ after $t = 0.5$ [s]. Even if we use more singular values, the amplitude would become larger and eliminate only small deviations.

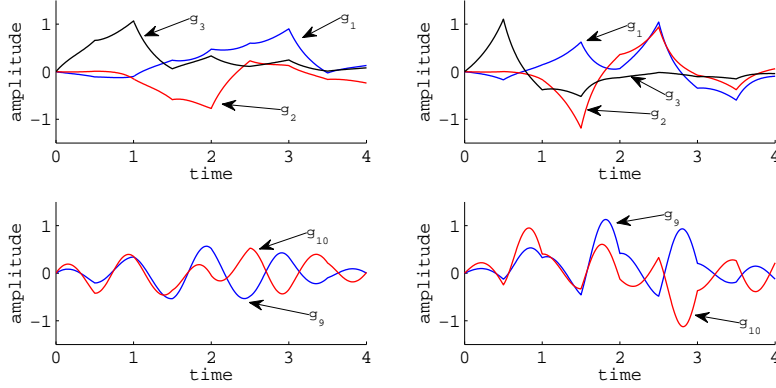


Figure 5: Singular vectors $g_i^1 \in L_2(0, h; \mathbb{R}^2) (\|g_i\|_Z = 1)$, left: x_1 , right: x_2 .

5. Conclusion

In the present paper, we have considered the SVD of an operator representing the input-output relation of a class of linear time-varying systems. The class of the systems describes time-driven switched linear systems. We have derived a computation method for singular values and singular vectors of the operator. Based on this SVD, the compensation signal design for switched linear systems with a periodic switching law is discussed. A numerical example was presented in order to demonstrate the fundamental properties of the proposed method.

Although we focus our attention on generating only a continuous-time signal to improve the transient, it would be important to design the switching signal and the continuous-time signal simultaneously. A possible extension of the research is incorporating system constraints, which has not been considered in this paper. From practical viewpoint, applying the method to a realistic model is also a topic of future research.

6. Acknowledgement

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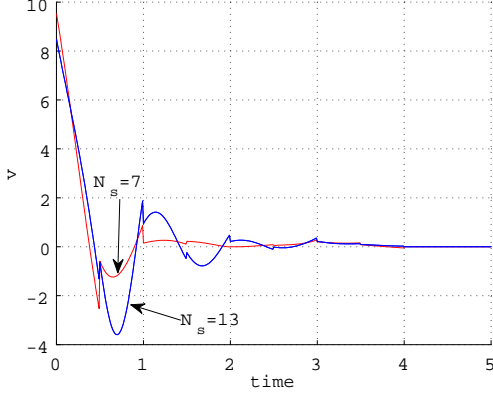


Figure 6: Compensation signal v .

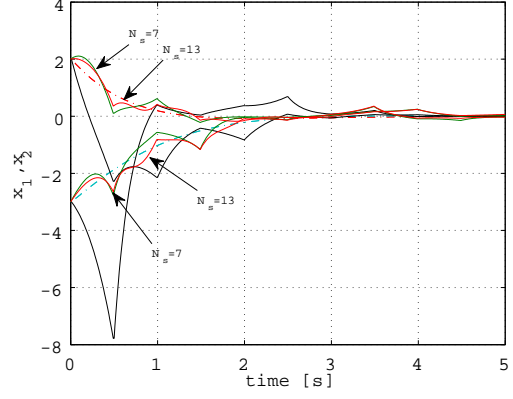


Figure 7: State response with compensation.

Appendix A. Derivation of the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ in (12)

First, we compute the following inner product:

$$\begin{aligned} \langle \hat{z}, \Gamma v \rangle_{\mathcal{Z}} &= \left\langle \begin{bmatrix} \hat{z}^0 \\ \hat{z}^1 \end{bmatrix}, \begin{bmatrix} (\Gamma v)^0 \\ (\Gamma v)^1 \end{bmatrix} \right\rangle_{\mathcal{Z}} \\ &= \hat{z}^{0T} (\Gamma v)^0 + \int_0^h \hat{z}^{1T}(\beta) (\Gamma v)^1(\beta) d\beta. \end{aligned} \quad (\text{A.1})$$

The first and second terms in (A.1) are calculated as follows:

(i) first term

$$\begin{aligned} \hat{z}^{0T} (\Gamma v)^0 &= \hat{z}^{0T} F \sum_{s=1}^{N-1} \Phi_{(N,s)}(h) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau \\ &\quad + \hat{z}^{0T} F \int_{t_{N-1}}^h e^{A_N(h-\tau)} B_N v(\tau) d\tau \\ &= \sum_{k=1}^{N-1} \int_{t_{k-1}}^{t_k} \left(B_k^T e^{A_k^T(t_k-\tau)} \Phi_{(N,k)}^T(h) F^T \hat{z}^0 \right)^T v(\tau) d\tau \\ &\quad + \int_{t_{N-1}}^{t_N} \left(B_N^T e^{A_N^T(h-\tau)} F^T \hat{z}^0 \right)^T v(\tau) d\tau, \end{aligned}$$

(ii) second term

$$\begin{aligned}
& \int_0^h \hat{z}^{1\Gamma}(\beta)(\Gamma v)^1(\beta)d\beta \\
&= \sum_{k=1}^{N-1} \sum_{s=1}^k \int_{t_k}^{t_{k+1}} \int_{t_{s-1}}^{t_s} \hat{z}^{1\Gamma}(\beta)E_k\Phi_{(k+1,s)}(\beta)e^{A_s(t_s-\tau)}B_s v(\tau)d\tau d\beta \\
&\quad + \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{\beta} \hat{z}^{1\Gamma}(\beta)E_k e^{A_k(\beta-\tau)}B_k v(\tau)d\tau \\
&= \sum_{k=1}^{N-1} \int_{t_{k-1}}^{t_k} \sum_{s=k}^{N-1} \int_{t_s}^{t_{s+1}} \hat{z}^{1\Gamma}(\beta)E_k\Phi_{(s+1,k)}(\beta)e^{A_k(t_k-\tau)}B_k d\beta v(\tau)d\tau \\
&\quad + \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\tau}^{t_k} \hat{z}^{1\Gamma}(\beta)E_k e^{A_k(\beta-\tau)}B_k d\beta v(\tau)d\tau.
\end{aligned}$$

For the transformation presented above, we used the reversal of the order of integration and the change in the order of summation. Consequently, by definition (2) of the inner product in \mathcal{V} , the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$, which satisfies $\langle \hat{z}, \Gamma v \rangle_{\mathcal{Z}} = \langle \Gamma^* \hat{z}, v \rangle_{\mathcal{V}}$, is given by (12). \square

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