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Modified Single-Loop-Single-Vector Method for Efficient Reliability-Based Design Optimization*

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Abstract

Single-Loop-Single-Vector (SLSV) method is to resolve the excessive computational cost problem in the reliability-based design optimization (RBDO) by decoupling the nested iteration loops. The key idea of the method is that the reliability constraint is transformed to the equivalent deterministic constraint by approximating the most probable point (MPP) using the point obtained from the previous iteration. However, the SLSV method sometimes suffers from numerical instability or inaccuracy problem. Thus, in this paper, a new modified SLSV method is proposed to improve its convergence capability effectively by utilizing Inactive Design and Active MPP Design together with modified-HMV (Hybrid Mean Value) method. The effectiveness of the proposed method is verified through some numerical examples.

Key words : Structural Reliability, Optimal Design, Reliability-Based Design Optimization, Numerical Stability, Modified SLSV Method

1. Introduction

For decades, there has been a great development in research field of reliability-based design optimization (RBDO) which considers an optimal design satisfying probabilistic constraints(1). The probabilistic constraints are evaluated inside of the optimization loop by the first order reliability method (FORM) that is a kind of minimization process(2). Therefore, the RBDO has been formulated as a nested iteration loop problem.

The reliability index approach (RIA) that is also called FORM is known as a conventional RBDO method that the probabilistic constraints are formulated by reliability index constraints that the value is evaluated by FORM inside of optimization iteration. The performance measure approach (PMA)(3), also known as fixed norm approach(4), was proposed to resolve the numerical instability problem of RIA as the nested iteration redloop. At the inner loop of PMA, the limit state function is maximized for equidistance hypersurface from the origin in the standardized normal distribution space (***U***-space), where the distance is equivalent to the target reliability index. At the outer loop, the objective function is minimized under the constraint that the maximum limit state function value should take a positive value. Though the inner loop does not evaluate the reliability, the optimum solution will satisfy the reliability constraints. Comparative studies of both approaches(5), (6) reported that PMA was superior to RIA in terms of numerical stability point of view.

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On the other hand, both RIA and PMA are formulated as a nested iteration loop problems. Thus, the high computational burden due to the nested iteration process could be a critical problem to be solved when applying this RBDO to practical use. Several numerical methods have been proposed to mitigate the computational burden in RBDO and they are mainly on the use of approximation methods(6), (7) or on the study of new algorithm trying to decouple the nested double loops of RBDO into a single loop.

Several computational efficient methods were proposed to reduce computational cost of the RBDO problem. Chen *et al.*(8) proposed a single loop method called SLSV(Single-Vector-Single-Loop) that enables single loop design optimization by replacing the reliability constraints to equivalent deterministic constraints by using an approximate design point (or most probable point (MPP)) information. Wu *et al.*(9) and Du *et al.*(10) proposed serial single loop methods called SFA (Safety Factor Approach) and SORA (Sequential Optimization and Reliability Assessment) which solve deterministic optimization after alternating the value of constraints or constraints shift by using MPP information obtained from reliability assessment. Yang and Gu(11) indicated that the SFA and SORA are conceptually identical, although they were presented independently. Also, Yang and Gu(11), (12) compared the computational efficiency and the reliability approximation accuracy of the SLSV, SFA and SORA and reported that SLSV was promising method in efficiency point of view.

This paper concentrates on numerical efficiency of the SLSV method. The method was originally formulated for normal distributed random variables. The idea was extended to be applied for non-normal distribution cases(13). As the SLSV has higher computational efficiency, the method has been applied to the reliability-based topology optimization problem(14), (15). On the other hand, Choi *et al.*(16) reported the SLSV method had a computational problem such that the method tends to be slow in the rate of convergence or to be divergent for concave function, especially for a large target reliability index.

Thus, in this paper, a new modified SLSV method is proposed. For the purpose, causes of the instability in the original SLSV are examined. Then, the proposed method is composed of modified-HMV (Hybrid Mean Value) method, Inactive Design(16) and Active MPP Design with the original SLSV method. Inactive Design corresponds to a deterministic optimum design, that is used as a starting point of RBDO. Active MPP Design corresponds to the sensitivity of the limit state function evaluated at the approximate MPP for the Inactive Design, that is used as an initial searching direction of RBDO. Then, the modified-HMV method is utilized to improve convergence property based on “Elimination of zigzagging iterations” method(5) developed for RIA or PMA. In this paper, the random variable is limited to normal distribution for simplicity. However, the proposed method in this paper resolving the convergence problem of the original SLSV method will be easily extended to non-normal distribution cases.

This paper is organized as follows. In Section 2, the original SLSV method is briefly described. Then, the modified SLSV method is explained in Section 3. Section 4 demonstrates the improvement of numerical efficiency and reliability approximation accuracy of the proposed method through numerical examples. Finally, conclusions are remarked.

Nomenclature

$\boldsymbol{d}$	design variables
$f(\boldsymbol{d})$	objective function
$g_j(\boldsymbol{d}, \boldsymbol{x})$	the j-th constraint function
NC	the number of constraints
ND	the number of design variables
NR	the number of random variables
$P[\cdot]$	probability function
$\boldsymbol{s}_j$	shifted vector for the j-th performance function

$\boldsymbol{X}$	random variables
$\boldsymbol{x}_j^*$	MPP for the j-th failure mode in $\boldsymbol{X}$-space
$\boldsymbol{\alpha}_j^*$	normalized gradient vector of the j-th failure mode evaluated at $\boldsymbol{x}_j^*$
β	reliability index
β_j	target reliability index for the j-th failure mode
$\boldsymbol{\mu}$	mean of random vector, $\boldsymbol{X}$
$\Phi(\cdot)$	standardized normal density function
σ_i	standard deviation of the i-th random variable
$\boldsymbol{\sigma}$	diagonal matrix composed of standard deviation, diag$[\sigma_i]$, $(i = 1, \cdots, NR)$
$\bullet^{(D)}$	superscript denoting deterministic optimum design
$\bullet^{(AD)}$	superscript denoting Active MPP design
$\bullet^{(ID)}$	superscript denoting Inactive design
$\bullet^{(k)}$	superscript denoting iteration number
$\bullet_{\mathrm{AMV}}$	subscript denoting Advanced Mean Value method
$\bullet_{\mathrm{CMV}}$	subscript denoting Conjugate Mean Value method
$\bullet_{\mathrm{HMV}}$	subscript denoting Hybrid Mean Value method

2. Reliability-Based Design Optimization

2.1. Formulation of RBDO

In this study, the RBDO problem [e.g.,(1)] is formulated to minimize the objective function subjected to the reliability constraints as follows:

$$\begin{aligned} \text{Minimize :} \quad & f(\boldsymbol{d}) \\ \text{subject to :} \quad & P[g_j(\boldsymbol{d}, \boldsymbol{X})] \leq 0] \leq \Phi(-\beta_j), \quad (j = 1, \cdots, NC) \\ & d_i^L \leq d_i \leq d_i^U, \quad (i = 1, \cdots, ND) \end{aligned} \tag{1}$$

where $\boldsymbol{d}$ is a design vector and $\boldsymbol{d}^L$ and $\boldsymbol{d}^U$ are lower and upper limits, respectively, and $\boldsymbol{x}$ indicates random variables. The problem has ND design variables and NR random variables. $g_j(\boldsymbol{d}, \boldsymbol{x})$ is the j-th limit state function that indicates the failure mode. The problem has NC reliability constraints. β_j is a target reliability index value of the corresponding failure mode, where $\Phi()$ is a standardized normal distribution function.

This paper concentrates on convergence problem of the SLSV method. Though the original SLSV method was extended to non-normal distributed random variables, this study focuses that the random variable $\boldsymbol{x}$ is restricted to independent normal distribution, $N(\boldsymbol{\mu}, \boldsymbol{\sigma})$. In addition, the mean value $\boldsymbol{\mu}$ of the random variable $\boldsymbol{x}$ is set as design vector $\boldsymbol{d}$ as in many researches. That is, the number of design variables ND is set to be identical to the number of random variables NR in this paper.

On the RIA (FORM), the MPP is described in $\boldsymbol{U}$-space that the random variables are transformed into the standardized normal distributed variables. When the random variables are independent normal distributed, the corresponding MPP in $\boldsymbol{X}$-space as shown in Fig. 1 is described as follows:

$$\boldsymbol{x}_j^* = \boldsymbol{\mu} - \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j^*, \qquad (j = 1, \cdots, NC) \tag{2}$$

$$\boldsymbol{\alpha}_j^* = \frac{\boldsymbol{\sigma} \nabla g_j\left(\boldsymbol{\mu}, \boldsymbol{x}_j^*\right)}{\left|\boldsymbol{\sigma} \nabla g_j\left(\boldsymbol{\mu}, \boldsymbol{x}_j^*\right)\right|}, \qquad (j = 1, \cdots, NC) \tag{3}$$

where $\boldsymbol{\sigma}$ is a diagonal matrix whose diagonal element consists of standard deviation of each random variable.

The reliability constraint is replaced as following deterministic constraint by utilizing that the MPP locates on the limit state surface ($g_j(\boldsymbol{\mu}, \boldsymbol{x}_j^*) = 0$):

$$g_j\left(\boldsymbol{\mu}, \boldsymbol{\mu} - \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j^*\right) \geq 0, \qquad (j = 1, \cdots, NC) \tag{4}$$

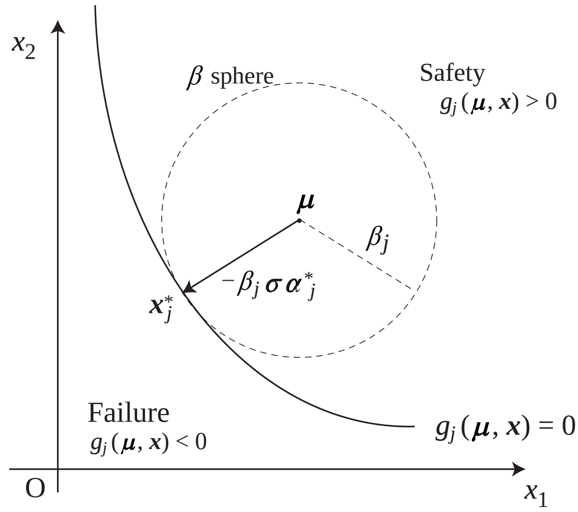


Fig. 1 Design variable and MPP in $\boldsymbol{x}$-space.

where the gradient vector $\boldsymbol{\alpha}_j^*$ in Eq. (3) should be evaluated at an MPP, $\boldsymbol{x}_j^*(= \boldsymbol{\mu} - \beta_j \boldsymbol{\sigma}\boldsymbol{\alpha}_j^*)$. However, the MPPs are not known in advance. Therefore, a conventional RBDO has been formulated as a nested iteration loop to find MPPs, $\boldsymbol{x}_j^*$.

2.2. Original SLSV Method

The key idea of SLSV method is to replace the normalized gradient vector $\boldsymbol{\alpha}_j$ by evaluating at MPP obtained in the previous optimization loop. The approximation makes the reliability constraints converted into the equivalent deterministic constraints and hence RBDO problem can be formulated as a single loop problem.

The computational flow is described as follows.

(1) Set the initial design $\boldsymbol{\mu}^{(k)}$ and the initial sensitivity $\boldsymbol{\alpha}_j^{(k)}$, where k is set to 0.

(2) Evaluate the limit state function $g_j(\boldsymbol{\mu}^{(k)}, \boldsymbol{x})$, where random vector $\boldsymbol{x}$ is set to $\boldsymbol{\mu} - \beta_j \boldsymbol{\sigma}\boldsymbol{\alpha}_j^{(k)}$. Then, normalized sensitivity vectors $\boldsymbol{\alpha}_j^{(k)}$ are obtained from $\nabla g_j(\boldsymbol{\mu}^{(k)}, \boldsymbol{x})$.

(3) Find the design candidate $\boldsymbol{\mu}^{(k+1)}$ by one dimensional search of the following optimization problem.

$$\begin{aligned} \text{Minimize}: \quad & f(\boldsymbol{\mu}^{(k+1)}) \\ \text{subject to}: \quad & g_j(\boldsymbol{\mu}^{(k+1)}, \boldsymbol{\mu}^{(k+1)} - \beta_j \boldsymbol{\sigma}_j \boldsymbol{\alpha}_j^{(k)}) \geq 0 \qquad (j = 1, \cdots, NC) \\ & \boldsymbol{d}^L \leq \boldsymbol{d} \leq \boldsymbol{d}^U \end{aligned} \tag{5}$$

After one-dimensional search, the random vector is set as follows:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{\mu}^{(k+1)} - \beta_j \boldsymbol{\sigma}_j \boldsymbol{\alpha}_j^{(k)} \tag{6}$$

(4) If the convergence criterion is satisfied, $\boldsymbol{d}^{(k+1)}$ is regarded as the optimum solution. Otherwise, increment k to $k+1$ and go back to Step (2) for continuing the next one-dimensional search.

The SLSV method can be applied for non-normal distribution cases[13].

The flowchart is illustrated in Fig. 2. Actually, constraint evaluation process does not need any sub-optimization loop. Instead, MPP is updated from $\boldsymbol{x}^{(0)}$ to $\boldsymbol{x}^*$ using the previous steepest descent direction while design variable vector converges from $\boldsymbol{\mu}^{(0)}$ to $\boldsymbol{\mu}^*$.

The equivalent deterministic constraints are discontinuously updated in every iteration. Consequently, the KKT necessary condition to be an MPP is not satisfied at intermediate MPP's in general. Anyway, by removing the inner loop of conventional RBDO for finding an exact MPP, computational efficiency of SLSV can be remarkably improved. However, it is known that SLSV may not converge or often lead to an inaccurate solution[16].

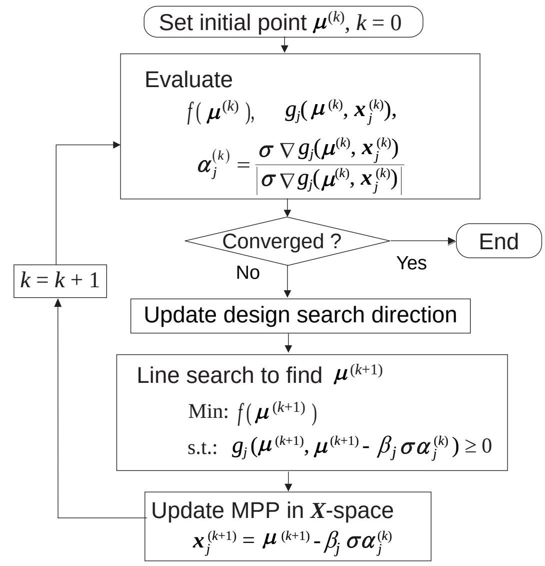


Fig. 2 Flowchart of Original SLSV Method[8]

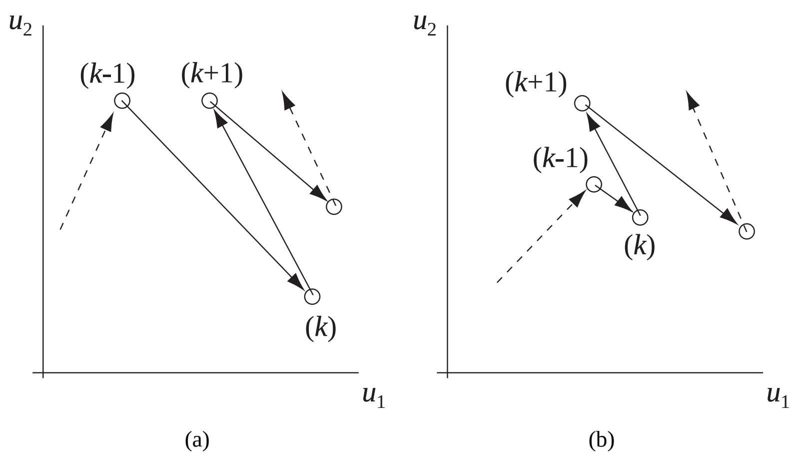


Fig. 3 Typical iteration types

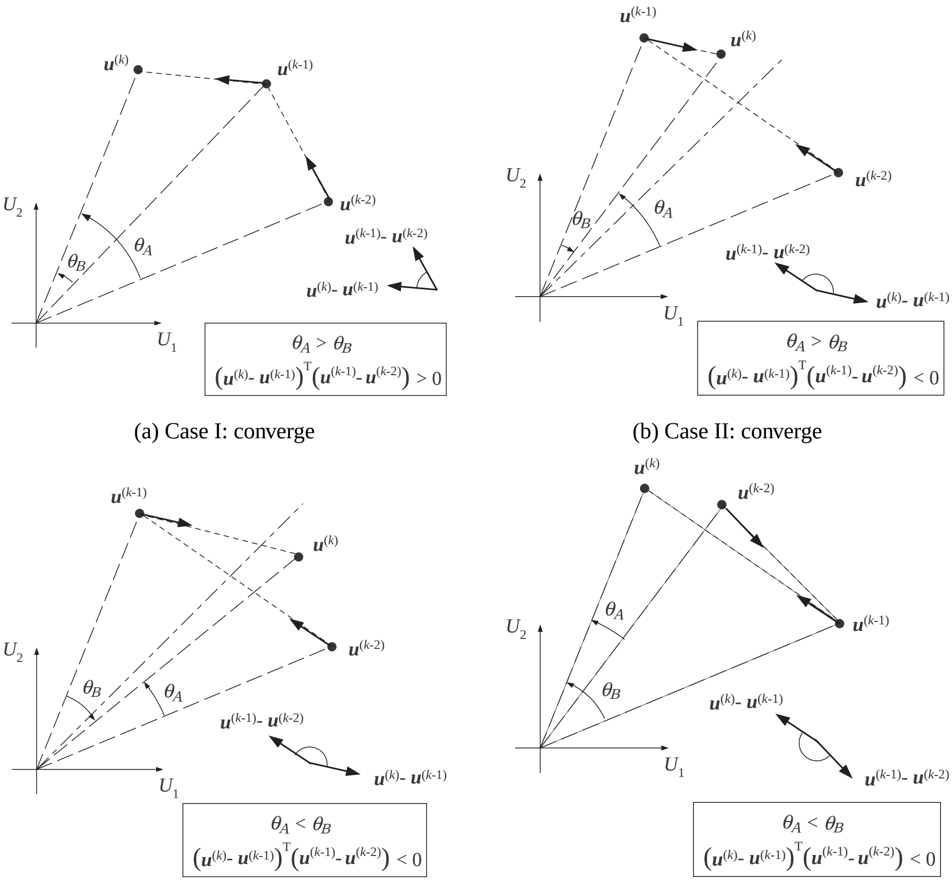
3. Modified SLSV Method

Convergence property of the SLSV method is improved by modified SLSV method that utilizes several efficient techniques, modified-HMV, Inactive Design and Active MPP Design.

3.1. MPP Update: Modified-HMV Method

One of the causes that make the numerical convergence difficult in SLSV method is the MPP update algorithm. According to Youn *et al.*[17], Eq. (6), known as AMV (Advanced Mean Value) method, has convergence problems for concave performance functions in PMA. The typical iteration types for convergence problems are shown in Fig. 3. One of the main reasons is that the searching direction corresponds to the steepest descent direction. In that cases, CMV (Conjugate Mean Value) method can resolve the convergence problems. The CMV method uses the average direction of the steepest directions of three consecutive iterations to determine the next searching direction as follows:

$$\begin{aligned} \boldsymbol{u}_{\mathrm{CMV}}^{(0)} &= \boldsymbol{0}, \\ \boldsymbol{u}_{\mathrm{CMV}}^{(1)} &= \boldsymbol{u}_{\mathrm{AMV}}^{(1)}, \\ \boldsymbol{u}_{\mathrm{CMV}}^{(2)} &= \boldsymbol{u}_{\mathrm{AMV}}^{(2)} \end{aligned} \qquad (7)$$



(a) Case I: converge

(b) Case II: converge

(c) Case III: slowly converge or diverge

(d) Case IV: diverge

Fig. 4 Typical successive iterations on MPP updates (in ***U***-space)

$$\boldsymbol{u}_{\mathrm{CMV}}^{(k)} = -\beta_t \frac{\boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-1)}) + \boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-2)}) + \boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-3)})}{\left|\boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-1)}) + \boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-2)}) + \boldsymbol{\alpha}(\boldsymbol{u}_{\mathrm{CMV}}^{(k-3)})\right|} \qquad \text{for} \quad k \geq 3 \tag{8}$$

where subscripts AMV and CMV correspond to each method and β_t is a target reliability index. Note that the method is formulated in ***U***-space.

Though the CMV method resolves the convergence problems for concave cases, the method deteriorates the convergence property for convex cases. Therefore, HMV (Hybrid Mean Value) method that selectively utilize AMV/CMV method for the types of the performance function was proposed. The type determined by following criterion that uses the steepest descent directions at the three consecutive iterations:

$$c^{(k+1)} = \left(\boldsymbol{\alpha}^{(k)} - \boldsymbol{\alpha}^{(k-1)}\right)^{\mathrm{T}} \left(\boldsymbol{\alpha}^{(k-1)} - \boldsymbol{\alpha}^{(k-2)}\right) \tag{9}$$

If $c^{(k+1)}$ is positive, the type is determined as convex, as CASE I in Fig. 4. Otherwise, the type is regarded as concave corresponding to either Case II, III or IV in Fig. 4. Then, the next iteration point is selected by applying Eq. (8).

In SLSV method, the concave type functions give rise to convergence problem. However, differently from the PMA case, convex functions can also show zigzagging phenomena since the mean values vary during MPP iteration. HMV method is still applicable for SLSV method because it selects AMV or CMV update according to the change of three consecutive gradient vectors in standard normal space.

As the similar method, Lee *et al.*[5] proposed "Elimination of zigzagging iteration" method independently for the double-loop method as RIA and PMA. The method is also applicable to improve numerical efficiency of the single loop method. Similar to HMV method, type of

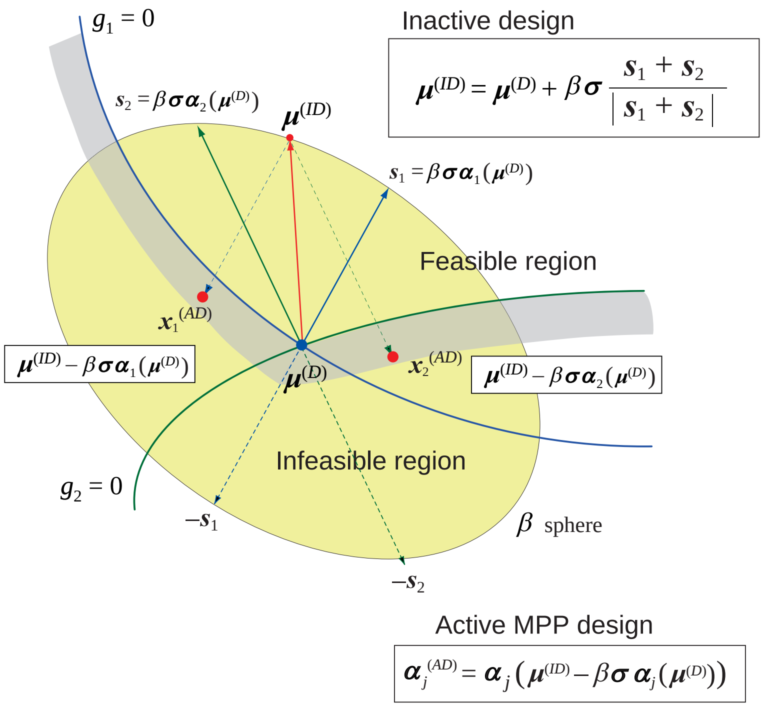


Fig. 5 Inactive design and active MPP design

the performance function is determined from the three successive points during iteration as follows:

$$\theta_A = \cos^{-1} \alpha^{(k-2)^{\mathrm{T}}} \alpha^{(k)}, \qquad (10)$$
$$\theta_B = \cos^{-1} \alpha^{(k-1)^{\mathrm{T}}} \alpha^{(k)}$$

If $\theta_A < \theta_B$, the continued iteration process is determined to suffer from the slow convergence or divergence. Then, the sensitivity is replaced by following equation:

$$\alpha^{(k)} = \alpha^{(k-2)} + \alpha^{(k-1)} \qquad (11)$$

This method is adopted to SLSV method and called Modified-HMV method in this paper.

3.2. Inactive Design

Although the modified-HMV method, in fact, is helpful in avoiding numerical divergence, Inactive Design and Active MPP Design discussed below should be selected as an initial design for stable convergence in SLSV method.

Inactive Design, proposed by Choi[16], is to move the initial design to inactive region where the probabilistic optimum is more likely to appear. The concept of Inactive Design is that the initial design is selected such that a deterministic optimum design $\boldsymbol{\mu}^{(D)}$ will correspond to the MPP of the initial design. The shifting vector $\boldsymbol{s}_j$ for the j-th performance function from $\boldsymbol{\mu}^{(D)}$ to the corresponding MPP is evaluated as follows:

$$\boldsymbol{s}_j = \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j \left(\boldsymbol{\mu}^{(D)}\right) = \beta_j \boldsymbol{\sigma} \frac{\boldsymbol{\sigma} \nabla g_j \left(\boldsymbol{\mu}^{(D)}\right)}{\left|\boldsymbol{\sigma} \nabla g_j \left(\boldsymbol{\mu}^{(D)}\right)\right|}, \qquad (j = 1, \cdots, NC) \qquad (12)$$

where the normalized gradient vector is evaluated at the mean value by Mean Value PMA (MV-PMA) method.

The Inactive Design $\boldsymbol{\mu}^{(ID)}$ is selected as average direction among active reliability constraint sets S_A with magnitude of the largest target reliability index value as follows:

$$\boldsymbol{\mu}^{(ID)} = \boldsymbol{\mu}^{(D)} + \left(\max_j \beta_j\right) \frac{\boldsymbol{\sigma} \sum_{j \in S_A} \boldsymbol{s}_j}{\left| \sum_{j \in S_A} \boldsymbol{s}_j \right|} = \boldsymbol{\mu}^{(D)} + \left(\max_j \beta_j\right) \frac{\boldsymbol{\sigma} \sum_{j \in S_A} \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j \left(\boldsymbol{\mu}^{(D)}\right)}{\left| \sum_{j \in S_A} \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j \left(\boldsymbol{\mu}^{(D)}\right) \right|} \quad (13)$$

Fig. 5 shows the initial MPP shift in case of two active constraints and the same target reliability indices. In RBDO, deterministic optimum has been typically used as an initial guess for a probabilistic optimum. Although this guess is logical and less risky than an arbitrary one, the influence of target reliability index or variation of variables can not be taken into account by this method. Inactive Design predicts a probabilistic optimum more precisely by considering probability-related information with only a few additional calculations. It is very useful, therefore, to reduce the computational cost in SLSV method as well as in double-loop methods like PMA and RIA.

3.3. Active MPP Design

However, it should be noted that the critical cause of instability in SLSV method lies on the fact that the form of constraints may discontinuously vary in every iterations, because the gradient vectors of the performance functions are updated based on the previous design. Therefore, the initial gradient vectors should be selected reasonably.

Active MPP Design proposed in this study is to use sensitivities of MPPs corresponding to the Inactive Design an initial sensitivity. At first, the approximated MPPs corresponding to Inactive Design $\mu^{(ID)}$ are obtained based on the gradient vector of the deterministic optimum design $\mu^{(D)}$ as follows:

$$\boldsymbol{x}_j^{(AD)} = \boldsymbol{\mu}^{(ID)} - \beta_j \boldsymbol{\sigma} \boldsymbol{\alpha}_j \left(\boldsymbol{\mu}^{(D)}\right) \quad (14)$$

The points are regarded as Active MPP Designs, $\boldsymbol{x}_j^{(AD)}$. Then, the normalized gradient vector of the performance function at Active MPP Designs are obtained as follows:

$$\boldsymbol{\alpha}_j^{(AD)} = \boldsymbol{\alpha}_j \left(\boldsymbol{x}_j^{(AD)}\right) \quad (15)$$

The positions of $\boldsymbol{x}_j^{(AD)}$ are also shown in Fig. 5.

3.4. Optimization flow of Modified SLSV Method

The modified SLSV method utilizes Inactive Design as an initial design of RBDO, and the normalized gradient vectors of performance functions evaluated at Active MPP Designs as the initial searching direction. During optimization, the modified-HMV method is adopted to modify the searching direction by using Eq. (11) to prevent from the wrong searching. Example of $\boldsymbol{\mu}^{(ID)}$, $\boldsymbol{x}^{(AD)}$ distributions in $\boldsymbol{X}$-space with deterministic and reliability-based optimum designs are shown in Fig. 6. Details of the problem formulation are described below in subsection 4.1.

The computational flow of the Modified SLSV method is described as follows.

(1) Evaluate the deterministic optimum design $\boldsymbol{\mu}^{(D)}$ and evaluate the normalized sensitivities $\boldsymbol{\alpha}_j^{(D)}$.

(2) Obtain Inactive Design $\boldsymbol{\mu}^{(ID)}$ in Eqs. (12) and (13), and set as the initial design, $\boldsymbol{\mu}^{(k)}$, where k is set to 0.

(3) Obtain Active MPP Designs $\boldsymbol{x}_j^{(AD)}$ in Eq. (14). Then, their normalized sensitivities $\boldsymbol{\alpha}_j^{(AD)}$ in Eq. (15) are set as the initial sensitivities, $\boldsymbol{\alpha}_j^{(k)}$.

(4) Evaluate the objective and the performance measure functions.

(5) If converged, $\boldsymbol{\mu}^{(k)}$ will be the optimum solution and terminated. Otherwise, continue to next step.

(6) Find the design candidate $\boldsymbol{\mu}^{(k+1)}$ by one dimensional search of the problem (5). Then, update the normalized sensitivities, $\boldsymbol{\alpha}_j^{(k+1)}$.

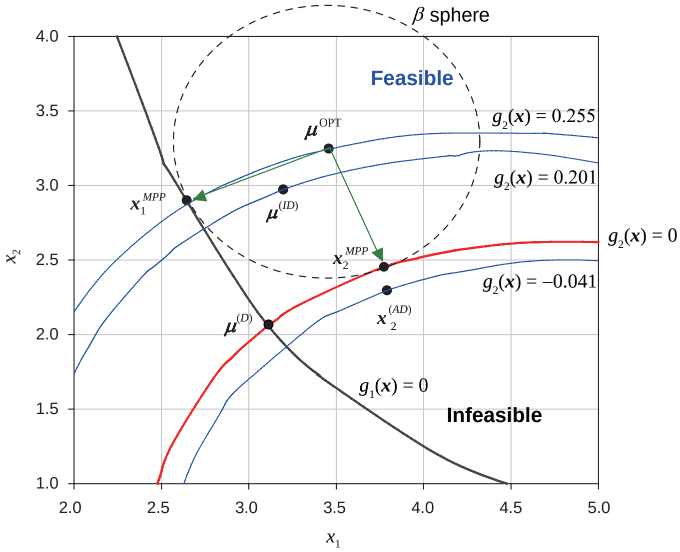


Fig. 6 Deterministic, inactive, active MPP and reliability-based designs in Example 1, ($\beta_j = 3, \sigma = 0.3$)

(7) Evaluate θ_{A_j} and θ_{B_j} in Eq. (10), if $k \geq 2$. Then, the sensitivity $\boldsymbol{\alpha}_j^{(k+1)}$ is modified according to Modified-HMV method in Eq. (11), if $\theta_{A_j} < \theta_{B_j}$.

(8) increment k to $k+1$ and go back to Step (4).

The flowchart of the proposed method is summarized in Fig. 7.

4. Numerical Examples

Numerical examples are given to demonstrate the effectiveness of the modified SLSV method in comparison with PMA, the original SLSV and so on. The first three examples are mathematical problems and the final is a mechanical design problem. All of the examples are known that the original SLSV method has numerical instability problem for slow convergence or divergence.

4.1. Example 1: Convergence Problem in the Original SLSV Method

The first example shows the convergence problem and its tendency in the original SLSV method. This example problem has a concave feasible region and is known as one that the original SLSV method has slow convergence or divergence problems for large target reliability indices[16), (17)]. The convergence problem and its tendency in the original SLSV method is investigated by the following two-dimensional mathematical RBDO problem[(6)]:

$$
\begin{aligned}
\text{Minimize :}\quad & f(\boldsymbol{d}) = d_1 + d_2 \qquad\qquad (16)\\
\text{subject to :}\quad & P(g_j(\boldsymbol{x}) \leq 0) \leq \Phi(-\beta_j^t), \quad (j = 1, 2)\\
\text{where :}\quad & g_1(\boldsymbol{x}) = \frac{x_1^2 x_2}{20} - 1 \leq 0\\
& g_2(\boldsymbol{x}) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1 \leq 0\\
& 0 \leq d_1 \leq 10, \quad 0 \leq d_2 \leq 10,
\end{aligned}
$$

where design variable $\boldsymbol{d} = \boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$ and the target reliability indices β_j vary from 2.0 to 5.0. Random variables $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ follow normal distribution, where the standard deviations are set as $\sigma = 0.3$ or $\sigma = 0.6$ (about 10-20% COV).

In this example, deterministic optimums are used as an initial design for all cases. Because when the center of design region, $(x_1, x_2) = (5, 5)$ was used as an initial design, all of the cases have diverged. The results of SLSV are listed in Table 1, where β_1^{HL} and β_2^{HL} are reliability indices for each constraint obtained by HL-RF method[(18)] to the final optimum

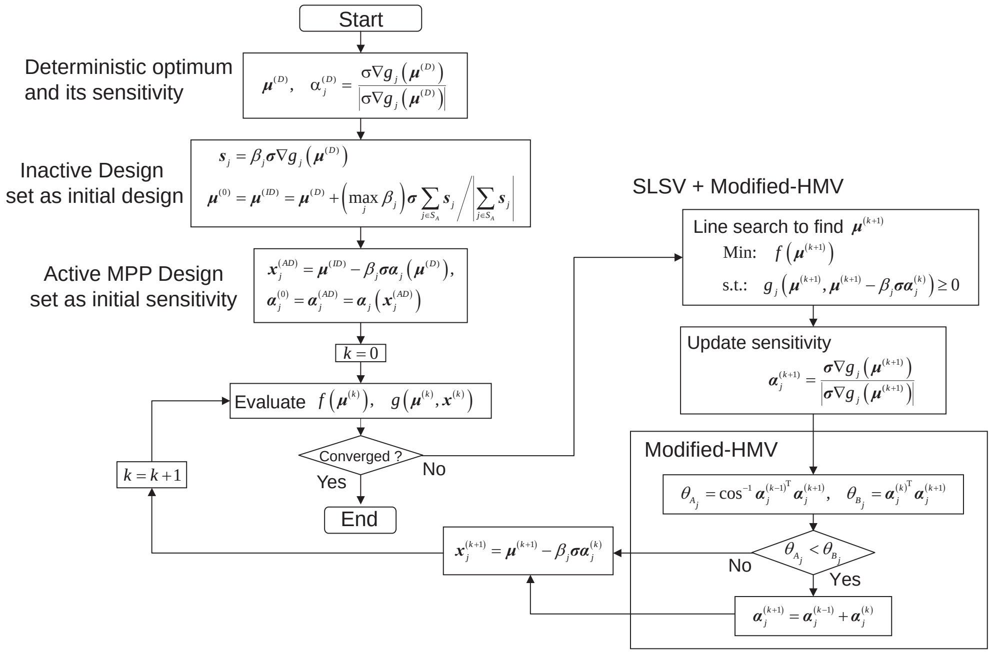


Fig. 7 Flowchart of modified-SLSV method

Table 1 Original SLSV method of Example 1

(a) Original SLSV, $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}^{(D)}, \sigma_j = 0.3$

β_t	Cost	d_1	d_2	NFE	β_1^{HL}	β_2^{HL}	%Err(1)	%Err(2)
2.0	6.1750	3.3155	2.8595	150	2.0009	1.8566	0.05	-7.17
3.0	6.7038	3.4571	3.2468	178	2.9997	2.8547	-0.01	-4.84
4.0	7.2556	3.6219	3.6336	150	4.0094	3.9057	0.24	-2.36
5.0	7.8097	3.8018	4.0080	120	4.9990	4.9686	-0.02	-0.63

(b) $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}^{(D)}, \sigma_j = 0.6$

β_t	Cost	d_1	d_2	NFE	β_1^{HL}	β_2^{HL}	%Err(1)	%Err(2)
2.0	7.2929	3.5978	3.6951	112	2.0039	2.0625	0.20	3.13
3.0	8.1655	4.6446	3.5209	116	3.4724	1.5010	15.75	-49.97
4.0				Diverge				
5.0				Diverge				

solution. These can be more intuitive measures to examine constraint satisfaction than $g_1(\mu^*)$ and $g_2(\mu^*)$ for an active constraint problem. NFE means the number of function evaluation required until achieving convergence and %Err means the percent error of β^{HL} against β^t.

Compared to the results of PMA listed in Table 2, SLSV method have advantages over PMA in computational efficiency as shown in Table 1. However, SLSV method may diverge or converge to a wrong solution for larger standard deviation or target reliability indices. Because the reliability-based optimum solution will be further from the deterministic optimum, the initial design to the reliability-based solution.

4.2. Example 2: Convex Nonlinear Performance Function

Computational efficiency of the proposed method is compared with several methods for the following two-dimensional mathematical problem with a convex nonlinear function cited from Youn *et al.*[17]:

$$\begin{aligned}
\text{Minimize :}\quad & f(\boldsymbol{d}) = 20 - d_1 + d_2 \qquad (17)\\
\text{subject to :}\quad & P(g(\boldsymbol{x}) \le 0) \le \Phi(-\beta)\\
\text{where :}\quad & g(\boldsymbol{x}) = -\exp(x_1 - 7) - x_2 + 10 \le 0\\
& 0 \le d_1 \le 10, \quad 0 \le d_2 \le 10\\
& \sigma_1 = \sigma_2 = 0.8
\end{aligned}$$

Table 2 PMA method of Example 1

(a) $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}^{(D)}, \sigma_j = 0.3$

β_t	Cost	d_1	d_2	NFE	β_1^{HL}	β_2^{HL}	%Err(1)	%Err(2)
2.0	6.1925	3.2951	2.8974	412	2.0004	1.9997	0.02	-0.01
3.0	6.7286	3.4365	3.2920	327	2.9997	3.0201	-0.01	0.67
4.0	7.2706	3.6074	3.6632	351	3.9998	4.0137	0.00	0.34
5.0	7.8160	3.7990	4.0171	466	4.9998	5.0002	0.00	0.00

(b) $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}^{(D)}, \sigma_j = 0.6$

β_t	Cost	d_1	d_2	NFE	β_1^{HL}	β_2^{HL}	%Err(1)	%Err(2)
2.0	7.2747	3.6052	3.6694	324	2.0003	2.0179	0.02	0.90
3.0	8.3807	3.9993	4.3814	360	3.0003	3.0352	0.01	1.17
4.0	9.4776	4.4496	5.0280	348	4.0002	4.0262	0.01	0.66
5.0	10.5608	4.9344	5.6264	591	5.0003	5.0059	0.01	0.12

Table 3 Comparison of SLSV, MV-PMA and PMA of Example 2 ($\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}^{(D)}$)

Method	Cost	d_1	d_2	NFE	β^{HL}	%Err
SLSV			Diverge			
MV-PMA	7.0881	5.9070	7.0049	100	2.5518	-14.94
PMA	7.3941	5.3003	7.0355	1056	3.0001	0.00

where design variable $\boldsymbol{d} = \boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$ and the target reliability index $\beta = 3.0$. Random variables $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ follow normal distribution, where the standard deviations are set as 0.8 (about 10-15% COV).

At first, the original SLSV method is compared with MV-PMA and PMA and the obtained result is listed in Table 3. In this case, the original SLSV method does not converge even though the deterministic optimum is used as a starting design. In addition, PMA converged to a correct optimum point, but MV-PMA converges to a wrong solution. The approximated MPP for the MV-PMA solution, $(\mu_1, \mu_2) = (5.9070, 7.0049)$ obtained by MV-PMA is compared with an exact MPP obtained by PMA in Fig. 8. Note that the figure is plotted in $\boldsymbol{U}$-space. The figure demonstrates the large approximation error in MV-PMA method.

Then, the computational efficiency and convergency of the modified-SLSV method is compared in Table 4, where, Modified-HMV method is denoted by "M-HMV", Inactive Design by "ID" and Active MPP Design by "AD". The results indicate that SLSV methods without Modified-HMV diverge, even if either Inactive Design, Active MPP Design or both Design is used. On the other hand, SLSV method with Modified-HMV converges to the almost identical design. However, the method without Inactive Design or Active MPP Design takes much more computational time than the methods with them. The number of function evaluations is only the half of PMA with Inactive Design that is a double loop method. It indicates that the modified-HMV method does not sufficiently avoid a zigzagging iterations, though the method overcome the divergence problem.

It is found that the proposed method with Inactive Design, Active MPP Design and modified-HMV method achieves high computational efficiency with sufficient accuracy. It indicates that the single-loop method should select an adequate initial design and sensitivities with avoiding a zigzagging iteration technique.

4.3. Example 3: Concave Performance Function

In the previous example, the modified-HMV method has no effect, when Inactive and Active MPP Designs are used. It is considered that effect of the modified-HMV method is significant for a concave function, not a convex function. For RIA or PMA, the computational efficiency is higher for a concave function to remove the zigzagging iteration[5]. The effect on a concave function is investigated in this example.

The following two dimensional mathematical problem is considered:

$$\text{Minimize}: \quad f(\boldsymbol{d}) = (d_1 + 2)^2 + (d_2 + 2)^2 - 2d_1 d_2 \tag{18}$$

$$\text{subject to}: \quad P(g(\boldsymbol{x}) \le 0) \le \Phi(-\beta)$$

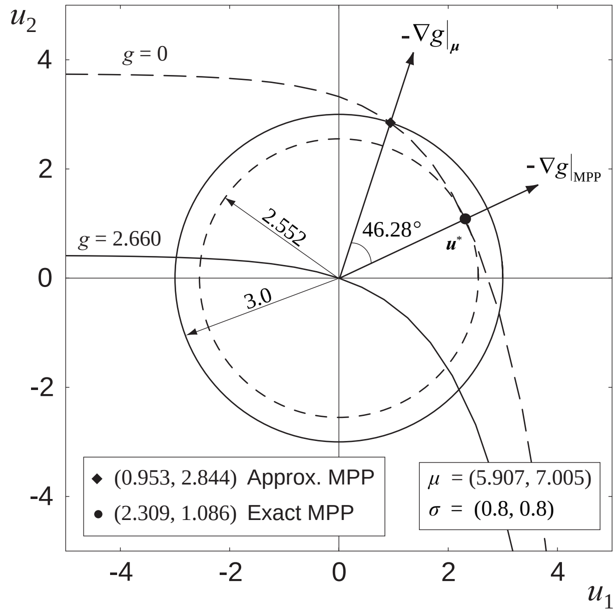


Fig. 8 Approximation error of MV-PMA Method in Example 2.

Table 4 RBDO Results of example 2

Method	Initial design	Cost	d_1	d_2	NFE	β^{HL}	%Err
SLSV	$\boldsymbol{\mu}^{(D)}$	Diverge					
SLSV + M-HMV	$\boldsymbol{\mu}^{(D)}$	7.340	5.276	7.327	363	2.9997	-0.01
SLSV + ID	$\boldsymbol{\mu}^{(ID)}$	Diverge					
SLSV + ID + M-HMV	$\boldsymbol{\mu}^{(ID)}$	Diverge					
SLSV + ID + AD	$\boldsymbol{\mu}^{(ID)}$	**7.394**	**5.301**	**7.305**	**42**	**2.9998**	**-0.01**
SLSV + ID + AD + M-HMV	$\boldsymbol{\mu}^{(ID)}$	**7.394**	**5.301**	**7.305**	**42**	**2.9998**	**-0.01**
PMA + ID	$\boldsymbol{\mu}^{(D)}$	7.394	5.299	7.307	751	3.0000	0.00

* $\boldsymbol{\mu}^{(D)} = (6.994, 9.005), \boldsymbol{\mu}^{(ID)} = (5.299, 7.307)$

$$\text{where} \ : \quad g(\boldsymbol{x}) = \frac{\exp(0.8x_1 - 1.2) + \exp(0.7x_2 - 0.6) - 5}{10}$$
$$0 \leq d_1 \leq 10, \quad 0 \leq d_2 \leq 10$$
$$\sigma_1 = \sigma_2 = 0.8$$

where the performance function is cited from(17). Random variables $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ follow normal distribution, where the standard deviations are set as 0.8. The target reliability index is set as $\beta = 3.0$.

The obtained results are summarized in Table 5, where notifications are the same as the previous example, Table 4. Deterministic design and reliability-based design are shown in Fig. 9. SLSV method without Active MPP Design diverge, or yield a wrong solution. Modified-HMV method with Active MPP Design required more function evaluations but higher reliability accuracy than that without Active MPP Design. However, the reliability approximation accuracy is higher The SLSV method Using Active MPP Design converge to near-optimum solution regardless of modified-HMV method.

4.4. Effect of Active MPP Design

Generally, convergence property of a nonlinear programming problem for a strongly nonlinear function depends on selection of the initial design. That is, if the starting point is selected close to an optimum point, the optimization easily find the optimum design. Inactive Design is regarded as such a strategy. However, Inactive Design alone is found to have little effect on the convergence property, especially for a concave performance function, even if the modified-HMV method is adopted.

Table 5 RBDO results of example 3

Method	Initial Design	Cost	d_1	d_2	NFE	β^{HL}	%Err
SLSV	$\boldsymbol{\mu}^{(D)}$			Diverge			
SLSV + M-HMV	$\boldsymbol{\mu}^{(D)}$	46.439	1.374	4.982	148	2.5815	-13.95
SLSV + ID	$\boldsymbol{\mu}^{(ID)}$			Diverge			
SLSV + ID + M-HMV	$\boldsymbol{\mu}^{(ID)}$			Diverge			
SLSV+ ID +AD	$\boldsymbol{\mu}^{(ID)}$	**40.722**	**4.045**	**4.133**	**94**	**2.9819**	**-0.60**
SLSV + ID + AD + M-HMV	$\boldsymbol{\mu}^{(ID)}$	**40.800**	**4.040**	**4.157**	**118**	**2.9994**	**-0.02**
PMA + ID	$\boldsymbol{\mu}^{(ID)}$	40.810	4.017	4.180	392	3.0018	0.06

* $\boldsymbol{\mu}^{(D)} = (1.909, 2.692), \boldsymbol{\mu}^{(ID)} = (2.873, 4.890)$

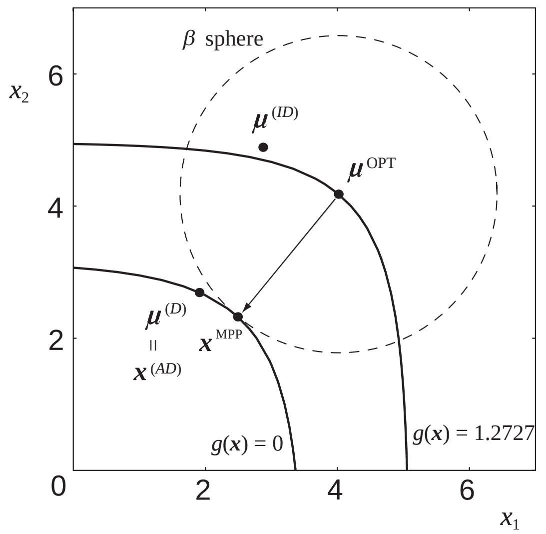


Fig. 9 Deterministic and reliability-based designs in Example 3

Active MPP Design is found to have an effect on convergence and reliability approximation from above examples. It is considered that the convergence property mainly depends on the change on directions of gradient vector of the performance function between the initial and the optimal design. Normalized gradient vectors of the performance function, $\boldsymbol{\alpha}^{(0)}$ are evaluated for the initial design candidate, $\boldsymbol{\mu}^{(D)}$, $\boldsymbol{\mu}^{(ID)}$ and $\boldsymbol{x}^{(AD)}$ as listed in Table 6. Note that $\boldsymbol{\mu}^{(D)}$ is equal to $\boldsymbol{x}^{(AD)}$ for Example 2 and 3 because of a single reliability constraint problem. However, they are different from each other for multiple constraint problem as Example 1.

In each point, angle between the sensitivity and that of the reliability-based optimum design is investigated. It is found that the angle between the initial and the optimum sensitivity is smaller regardless of the function types, when the Active MPP Design is adopted. In the Active MPP Design case, all of the optimization processes converge stably. On the other hand, for cases of the larger angles, the optimization processes diverge or converge slowly. That is, Active MPP Design has an effect of approximating the initial sensitivity direction to the optimum sensitivity direction and hence, the convergence property is improved.

4.5. Example 4: Ten-bar truss design problem

The effect of the proposed method is applied to a Ten-bar truss design problem shown in Fig. 10. This truss structure has been used as an example in many literatures on optimization, [i.e.,(19)] and also RBDO. In this study, the volume minimization problem subject to only the stress reliability constraint is considered(20). The member cross-sectional area and the allowable stresses are adopted as normally distributed random variables, and the mean values of the cross-sectional area are treated as design variables. The reliability-based optimization problem is formulated as follows:

$$\text{Minimize}\ :\quad F(\boldsymbol{d}) = \sum_{i=1}^{10} d_i l_i \tag{19}$$

Table 6 Convergence Improvement with Active MPP Design

	$\alpha^{(0)}$ Calculation point		$\alpha^{(0)}$	Relative angle(deg)*
Ex. 1 (g_1)	$\mu^{(D)}$	(3.103, 2.078)	(0.801, 0.598)	12.4
	$\mu^{(ID)}$	(3.197, 2.973)	(0.881, 0.474)	3.9
Convex	$x^{(AD)}$	(2.476, 2.434)	(0.891, 0.453)	2.6
	x^*	(2.626, 2.990)	(0.991, 0.413)	
Ex.1 (g_2)	$\mu^{(D)}$	(3.103, 2.078)	(-0.659, 0.752)	21.3
	$\mu^{(ID)}$	(3.197, 2.973)	(-0.396, 0.918)	3.3
Concave	$x^{(AD)}$	(3.790, 2.296)	(-0.383, 0.924)	2.6
	x^*	(3.773, 2.476)	(-0.342, 0.940)	
Ex. 2	$\mu^{(D)}$	(6.994, 9.005)	(-0.705, -0.709)	0.4
	$\mu^{(ID)}$	(5.299, 7.307)	(-0.180, -0.984)	34.9
Convex	$x^{(AD)}$	(6.994, 9.005)	(-0.705, -0.709)	0.4
	x^*	(7.008, 8.992)	(-0.710, -0.704)	
Ex. 3	$\mu^{(D)}$	(1.909, 2.692)	(0.402, 0.916)	18.4
	$\mu^{(ID)}$	(2.873, 4.890)	(0.200, 0.980)	30.6
Concave	$x^{(AD)}$	(1.909, 2.692)	(0.402, 0.916)	18.4
	x^*	(2.490, 2.324)	(0.670, 0.742)	

(*) Angle between $\alpha^{(0)}$ and α^*, or, $|\Delta\alpha|_{OPT}$

Table 7 Random variables of ten-bar truss problem

	Mean	Standard deviation
Area x_i, $(i = 1, \cdots, 10)$ (in^2)	d_i	$0.05d_i$
Strength R (psi)	2.5×10^4	1.25×10^3

$$\begin{aligned} \text{subject to :} \quad & P(g_j(\mathbf{x}, R) \leq 0) \leq \Phi(-\beta_j), \qquad (j = 1, \cdots, 10) \\ \text{where :} \quad & g_j(\mathbf{x}, R) = R - |s_j| \leq 0, \qquad (j = 1, \cdots, 10) \\ & 0.1 \leq d_i \leq 10 \qquad (i = 1, \cdots, 10) \\ & \beta_j = 2.0 \qquad (j = 1, \cdots, 10) \end{aligned}$$

where and l_i and s_i are the member length and the stress of the i-th member, respectively. d_i is the mean value of cross-sectional area of i-th member as design variables. x_i and R are random variables with normal distribution as listed in Table 7. The target reliability index is set as $\beta_j = 2.0$ for all members in order to compare with Yeun *et al.*s' results[20].

The comparative results are shown in Table 8, where RSM and PGP indicate the response surface method and polynomial genetic programming, respectively,[20] that both methods require sampling. NIT and NFE indicate the number of iterations and the number of function evaluations, respectively. The original SLSV method in the fourth column shows large errors compared to that of PMA, because of the discontinuous constraint charges. On the other hand, the modified SLSV method has noticeable improvement in accuracy compared to that of original SLSV. From numerical efficiency point of view, the method also gives relatively good result as well as the PGP (Polynomial Genetic Programming) + RSM(Response Surface Method) with high fidelity model case, although no approximation technique is applied in the proposed method.

5. Conclusions

SLSV method, proposed as an efficient RBDO methodology, actually shows good computational efficiency compared to traditional double-loop methods. However, the discontinuous constraint change between iterations makes the method unstable often resulting in divergence or inaccurate solution. Therefore, a methodology that can complement the convergence capability of SLSV method has been needed.

This paper has fully examined the causes of instability in SLSV method and proposed a Modified-SLSV method comprises of Modified-HMV, Inactive Design and Active MPP

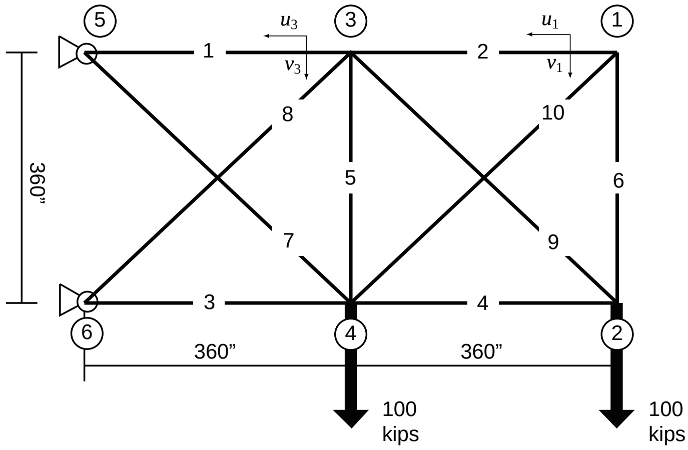


Fig. 10 Ten-bar truss design problem

Table 8 RBDO results of ten-bar truss design problem

		ID*+PMA	PGP+RSM (LF)*(20)	PGP+RSM (HF)**(20)	SLSV	Modified SLSV
Objective		**1840**	1830.8	**1841.3**	1875.4	**1844.5**
Optimum	1	**9.192**	9.145	**9.195**	10.00	**9.223**
	2	**0.100**	0.100	**0.100**	0.101	**0.100**
	3	**9.316**	9.270	**9.319**	9.329	**9.323**
	4	**4.567**	4.544	**4.569**	4.633	**4.569**
	5	**0.100**	0.100	**0.100**	0.110	**0.100**
	6	**0.100**	0.100	**0.100**	0.106	**0.100**
	7	**6.615**	6.583	**6.629**	6.600	**6.616**
	8	**6.441**	6.407	**6.445**	6.508	**6.478**
	9	**6.457**	6.424	**6.459**	6.456	**6.479**
	10	**0.100**	0.100	**0.100**	0.104	**0.100**
NIT		**4**	(54 samples)	**(54+10 samples)**	17	**22**
NFE		**14547**	1242	**3672**	4267	**5637**
NFE Ratio		**100%**	8.54%	**25.24%**	29.30%	**38.80%**

ID: Inactive Design, LF: Low Fidelity Model, HF: High Fidelity Model

Design with the original SLSV method. Numerical examples were presented for verification purpose and the proposed method is proved to be working for problems that have reasonable range of variation and target reliability.

However, it should be noted that the proposed method does not fundamentally remove the causes of instability but try to decrease the errors causing divergence, that is indicated in Youn *et al.*s' research(21). In other words, the proposed method does not mathematically guarantee numerical convergence in SLSV, but supply a much better chance of convergence for practical use.

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