Examples of Algebraic Varieties with Kobayashi Hyperbolicity

EXAMPLES OF ALGEBRAIC VARIETIES WITH KOBAYASHI HYPERBOLICITY

Takanori MATSUNO*

Abstract

Many interesting examples of hyperbolic hypersurfaces in the complex projective space $\mathbf{P}^3(\mathbf{C})$ have been known. In this paper, we give some examples of 2-dimensional hyperbolic algebraic varieties which are defined as intersections of Fermat varieties in $\mathbf{P}^4(\mathbf{C})$.

Key Words : Kobayashi hyperbolicity, Nevanlinna theory

1 Introduction

In [K], Kobayashi asked whether a generic hypersurface in the complex projective space $\mathbf{P}^n(\mathbf{C})$ of degree enough large with respect to n is hyperbolic or not. This conjecture is true for $n = 2$. In fact, a plane curve with genus greater than or equal to 2, does not admits no non-constant holomorphic mapping from $\mathbf{C}$, because its universal covering space is a ball. For $n \geq 3$ this problem is still open. But there have been known many examples of hyperbolic hypersurfaces in $\mathbf{P}^3(\mathbf{C})$ ([Br-Gr][D][F2][Go][N][S1][S2]). In this paper, we give some examples of 2-dimensional hyperbolic algebraic varieties which are defined as intersections of Fermat varieties in $\mathbf{P}^4(\mathbf{C})$.

2 Preliminaries

We recall some definitions and a result.

Definition 2.1. For two entire functions f and g which are not identically zero, we say they are equivallent if there exists a constant c $(c \neq 0)$ such as $f = cg$ holds. This introduces an equivalence relation in each set of entire functions which are not identically zero. We mean by the notation $f \sim g$ that f and g are equivalent.

Definition 2.2. Let f be a holomorphic mapping of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$. A representation $\tilde{f} = (f_0, \dots, f_n)$ of f is a holomorphic mapping of $\mathbf{C}$ into $\mathbf{C}^{n+1}$ such that $\tilde{f}^{-1}(\mathbf{0}) \neq \mathbf{C}$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ for each $z \in \mathbf{C} \setminus \tilde{f}^{-1}(\mathbf{0})$, where $(X_0 : \cdots : X_n)$ is a homogeneous coordinate system. A representation $\tilde{f}$ is called to be reduced if $\tilde{f}^{-1}(\mathbf{0}) = \phi$.

(Received April 11, 2007)

* Dept. of Industrial Systems Engineering : Natural Science

The following theorem was given by Green [Gr] and Fujimoto [F1]:

Theorem 2.1. *Let $f_0, \dots, f_n$ be entire functions which are not identically zero such that $f_0^d + \cdots + f_n^d \equiv 0$, where d is a positive integer. If $d \geq n^2$, then*

$$\sum_{f_j \in I} f_l^d \equiv 0$$

for each equivalence class I of $\{f_0, \dots, f_n\}$. Especially each class has at least two elements.

3 Main theorem

Let d be a positive integer. Put $M_d := \{X_0^d + \cdots + X_4^d = 0\}$, where $X_0, \dots, X_4$ are homogeneous coordinates, which is a Fermat varietiy of degree d in $\mathbf{P}^4(\mathbf{C})$. First take d as d is greater than or equal to 16. And then take d' such as the set of d-th roots of -1 and that of the d'-th roots of -1 do not share any element. We define a complex surface S in $\mathbf{P}^4(\mathbf{C})$ as $S := M_d \cap M_{d'}$. Then we have the following theorem:

Theorem 3.1. *S is Kobayashi hyperbolic.*

Proof. Assume that there exists a holomorphic mapping f of $\mathbf{C}$ into $\mathbf{P}^4(\mathbf{C})$ with reduced representation $\tilde{f} = (f_0, f_1, f_2, f_3, f_4)$ such that $f(\mathbf{C}) \subset S$. Since $f(\mathbf{C}) \subset M_d$, $f_0^d + \cdots + f_4^d \equiv 0$.

First we assume that each f_j is not identically zero. By **Theorem 2.1**, the set $\{f_0, \dots, f_4\}$ of entire functions can be divided into each equivalence classes. Let N be the number of elements of an equivalence class of f_0.

(I) The case that $N = 5$. Cleary, f is constant in this case.

(II) The case that $N = 4$. This case cannot occur. Because each equivalent class has at least two elements.

(III) The case that $N = 3$. In this case, say, $f_1 = c_1 f_0$, $f_2 = c_2 f_0$, $f_3 = c_3 f_4$. By **Theorem 2.1**, we get $1 + c_1^d + c_2^d = 0$ and $c_3^d + 1 = 0$. But by the assumption for d', this does not satisfy the conditon that $f(\mathbf{C}) \subset M_{d'}$. Then this case cannot occur.

(IV) The case that $N = 2$. This case is as same as the above. Then this case cannot occur.

(V) The case that $N = 1$. This case cannot occur. Because each equivalent class has at least two elements.

Next we consider the case that $f_j \equiv 0$ for some j. In the case $f_j \equiv 0$ for only one j, the each equivalence class has 4 or 2 elements. If an equivalence class has 4 elements, it is clear that f is constant. If each equivalence class has 2 elements, the conditon that $f(\mathbf{C}) \subset M_{d'}$ is not satisfied, same as above (III). Then this case cannot occur. In the case that $f_j \equiv 0$ for more than 2 $j's$, it is clear that f is constant since the image $f(\mathbf{C})$ is included in a hyperbolic Riemann surface.

We have shown that every holomorphic mapping f of $\mathbf{C}$ into S is constant. So S is hyperbolic. The proof is completed.

For example, if we take d such as d is even and is greater than or equal to 16 and put $d' = 1$, then S is biholomorphic to the hypersurface $S_d := \{X_0^d + X_1^d + X_2^d + x_3^d + (-X_0 - X_1 - X_2 - X_3)^d = 0\}$ in $\mathbf{P}^3(\mathbf{C})$. Then we have:

Theorem 3.2. *S_d is a hyperbolic hypersurface in $\mathbf{P}^3(\mathbf{C})$.*

Remark. This example of a hyperbolic hypersurface is not a new one. This is given in a rather complicated situation in [S1].

References

[Br-Gr] R. Brody and M. Green : *A family of smooth hyperbolic hypersurfaces in* $\mathbf{P}^3$, Duke Math. J., 44(1977), 873-874

[D] J. P. Demailly : *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Proc. Sympos. Pure Math., Vol. 62, Part2, Amer. Math. Soc., Providence, RI, 1997, 285-360.

[F1] H. Fujimoto : *On meromorphic maps into the complex projective space*, J. Math. Soc. Japan, 26(1974), 272-288.

[F2] H. Fujimoto : *A family of hyperbolic hypersurfaces in the complex projective space*, The Chuang special issue. Complex Variables Theory Appl., 43(2001), no. 3-4, 273-283.

[Go] J. Goul : *Algebraic families of smooth hyperbolic hypersurfaces of low degree in* $\mathbf{P}^3_{\mathbf{C}}$, manuscripta math., 90(1996), 521-532.

[Gr] M. L. Green : *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math., 97(1975), 43-75.

[K] S. Kobayashi : *Hyperbolic Manifolds and Holomorphic Mappings* , Dekker, 1987.

[N] A. Nadel : *Hyperbolic surfaces in* $\mathbf{P}^3$, Duke Math. J., 58(1989), 749-771

[S1] M. Shirosaki : *Hyperbolic hypersurfaces in the complex projective spaces of low dimensions*, Kodai Math. J., 23(2000), 224-233.

[S2] M. Shirosaki : *A hyperbolic hypersurface of degree 10*, Kodai Math. J., 23(2000), 376-379.