# A Remark on Maximal Galois Coverings over the Complex Projective Plane



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# A REMARK ON MAXIMAL GALOIS COVERINGS OVER THE COMPLEX PROJECTIVE PLANE

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**Abstract**

We set $C = \{(x,y) \in \mathbf{C}^2 | y^2 = x^3\}$ and put $D = e\overline{C} + mL_\infty$, where $\overline{C}$ is the closure of $C$ in the complex projective plane $\mathbf{P}^2(\mathbf{C})$ and $L_\infty$ is the line at infinity. In this short note we shall prove that if $e \geq 6$ and $m = 2$, then there does not exist a maximal Galois covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$.

## 1 Introduction

We set $C = \{(x,y) \in \mathbf{C}^2 | y^2 = x^3\}$ and we denote by $\overline{C}$ its closure in $\mathbf{P}^2(\mathbf{C})$ and by $L_\infty$ the line at infinity of $\mathbf{P}^2(\mathbf{C})$. Let $e, m$ be positive integers and consider the effective divisor $D = e\bar{C} + mL_\infty$ on $\mathbf{P}^2(\mathbf{C})$. For this type of the divisor the following results are known ( cf. [3], [4] and [5] ):

**Theorem 1.1** *Suppose that $e = 2, 3, 4$ or $5$ and let $D = e\overline{C} + mL_\infty$. Then there exists a maximal Galois covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$ if and only if $(e,m) =$ $(2,2), (3,1), (3,2), (3,4), (4,2), (4,4), (4,8),$ $(5,2), (5,4), (5,5), (5,10), (5,20)$.*

**Theorem 1.2 (Namba[5])** *Suppose $e = 6$ and $m = 2$. Then there does not exist a maximal Galois covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$.*

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We modify the idea of his proof for Theorem1.2 and extend his result. In this note we shall prove :

**Theorem 1.3** *Suppose $e \geq 6$ and $m = 2$. Then there does not exist a maximal Galois covering $\pi : X \to \mathbf{P}^2(\mathbf{C})$ which branches at $D$.*

## 2 Summary of the theory of maximal coverings

Let $M$ be an $n$–dimensional complex manifold and $D = \sum_{i=1}^{s} e_i D_i$ $(e_i > 0)$ an effective divisor on $M$, where $D_i$ is an irreducible component of $D$.

**Definition 2.1** *Let $\pi : X \to M$ be a Galois covering which branches at $D$. We say that $\pi$ is the maximal Galois covering of $(M, D)$ if $\pi$ dominates any branched covering over $M$ which branches at most at $D$.*

We set $B = D_1 \cup \ldots \cup D_s$. We fix a base point $p_0 \in M - B$ and take a point $p_i \in D_i - Sing(B)$, where $Sing(B)$ is the singular locus of $B$. Let $W$ be a small open neighborhood of $p_i$

with a coordinate system $(w_1, \ldots, w_n)$ such that $p_i = (0, \ldots, 0)$ and $D_i \cap W = \{(w_1, \ldots, w_n) \in W | w_n = 0\}$. We define a loop $\alpha_i$ as $\alpha_i = (0, \ldots, 0, \varepsilon e^{2\pi\sqrt{-1}t})$ $(0 \leq t \leq 1)$, where $\varepsilon > 0$ is a sufficiently small positive number. $\alpha_i$ rounds $D_i$ once in a counterclockwise direction with $p_i$ as a center. We take a path $\beta_i$ in $M - B$ from $q_i = (0, \ldots, 0, \varepsilon)$ to $p_0$ and we define $\gamma_i = \beta_i^{-1}\alpha_i\beta_i$ which is a loop in $M - B$ with $p_0$ as a starting point. Let $J = \ll \gamma_1^{e_1}, \ldots, \gamma_s^{e_s} \gg$ be the smallest normal subgroup of $\pi_1(M - B, p_0)$ which contains $\gamma_1^{e_1}, \ldots, \gamma_s^{e_s}$.

Consider the following condition on a subgroup $K$ of $\pi_1(M - B, p_0)$:

**Condition 2.2** *Let $K$ be a subgroup of $\pi_1(M - B, p_0)$ such that $J \subset K$. If $\gamma_j^d \in K$, then $d \equiv 0$ (mod $e_j$) for $(1 \leq j \leq s)$.*

For any singular point $p \in Sing(B)$, take $V_p$ a sufficiently small open neiborhood of $p$ in $M$ and let $i_p : V_p - B \to M - B$ be the inclusion map and $(i_p)_* : \pi_1(V_p - B, p'_0) \to \pi_1(M - B, p'_0)$ be the homomorphism induced by $i_p$, where $p'_0$ is a fixed point in $V_p - B$. Let $\beta$ be a path in $M - B$ from $p'_0$ to $p_0$. Then we have the isomorphism

$$\pi_1(M - B, p'_0) \ni \gamma \mapsto \beta^{-1}\gamma\beta \in \pi_1(M - B, p_0).$$

We identify $\pi_1(M - B, p'_0)$ with $\pi_1(M - B, p_0)$ through this isomorphism.

**Condition 2.3** *Let $K$ be a subgroup of $\pi_1(M - B, p_0)$ such that $J \subset K$. For any singular point $p \in Sing(B)$, $K_p := (i_p)_*^{-1}(K)$ is a subgroup of $\pi_1(V_p - B, p'_0)$ of finite index.*

We put $\widetilde{K} := \cap K$, where $\cap$ runs over all subgroups $K$ of $\pi_1(M - B, p_0)$ with $J \subset K$ which satisfy Condition2.3. Then $\widetilde{K}$ is a normal subgroup of $\pi_1(M - B, p_0)$ such that $J \subset \widetilde{K}$.

We put $\widetilde{G} = \pi_1(M - B, p_0)/\widetilde{K}$.

**Theorem 2.4 (Namba[5])** *There is a maximal covering $\tilde{\pi} : \widetilde{M}(D) \to M$ which branches at $D$ if and only if $\widetilde{K}$ satisfies both Condition2.2 and 2.3. In this case, (1) $\tilde{\pi} : \widetilde{M}(D) \to M$ corresponds to $\widetilde{K}$ and so the Galoi group $Gal(\tilde{\pi}) = \widetilde{G}$ and (2) $\widetilde{M}$ is simply connected.*

## 3 Proof of the Theorem

From Zariski-van Kampen theorem we have the following presentation of the fundamental group $\pi_1(\mathbf{P}^2 - \{\overline{C} \cup L_\infty\})$ of the complement of $\overline{C} \cup L_\infty$ in $\mathbf{P}^2(\mathbf{C})$

**Lemma 3.1** $\pi_1(\mathbf{P}^2 - \{\bar{C} \cup L_\infty\}) = \pi_1(\mathbf{C}^2 - C) = < \gamma_1, \gamma_2, \delta \,|\, \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2, \delta = (\gamma_1\gamma_2\gamma_1)^{-1} >$

Let $G(2,3,e) = < S, T | S^2 = T^3 = (ST)^e = 1 >$ be the Schwartz triangular group.

We define $G_e = < a, b | a^e = b^e = 1, aba = bab >$ which is isomorphic to $\pi_1(\mathbf{C}^2 - C)/\ll a^e, b^e \gg$, where $\ll a^e, b^e \gg$ is the smallest normal subgroup in $\pi_1(\mathbf{C}^2 - C)$ which contains $a^e, b^e$.

**Lemma 3.2** *There is the following exact sequence*

$$1 \to < z > \to G_e \xrightarrow{f} G(2,3,e) \to 1$$

*where $< z >$ is the cyclic group generated by $z = (aba)^2$ and the hommomorphism $f$ is defined by $f(a) = ST$ and $f(b) = TS$.*

Let $J_{e,m} = \ll \gamma_1^e, \gamma_2^e, \delta^m \gg$ be the smallest normal subgroup of $\pi_1(\mathbf{P}^2 - \{\bar{C} \cup L_\infty\})$ which contains $\gamma_1^e, \gamma_2^e, \delta^m$. Let $G_{e,m} = \pi_1(\mathbf{P}^2 - \{\bar{C} \cup L_\infty\})/J_{e,m}$ which has a presentation $< \alpha, \beta | \alpha^e = \beta^e = \delta^m = 1, \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} >$.

We have a surjective homomorphism $g : G_e \to G_{e,m}$ by sending $g(a) = \alpha$ and $g(b) = \beta$. We assume $m$ is even. Then we have

**Lemma 3.3** *The kernel of $g$ is $< z^{m/2} >$, where $< z^{m/2} >$ is a cyclic group generated by $z^{m/2}$.*

Hence we have the following exact sequence:

$$1 \to < z^{m/2} > \to G_e \xrightarrow{g} G_{e,m} \to 1.$$

Combining above exact sequences, we have the following exact sequence:

$$1 \to < z > / < z^{m/2} > \to G_{e,m} \to G(2,3,e) \to 1.$$

So in particular, if $m = 2$, then

$$G_{e,2} \simeq G(2,3,e).$$

Let $\mathbf{P}^1$ be the complex projective line and take three points, say $p_1$, $p_2$ and $p_3$. We choose a point $p_0 \in \mathbf{P}^1 - \{p_1, p_2, p_3\}$ as a base pont. It is well known that the fundamental group $\pi_1(\mathbf{P}^1 - \{p_1, p_2, p_3\}, p_0)$ has a following presentation $< \gamma_1, \gamma_2, \gamma_3 \mid \gamma_3\gamma_2\gamma_1 = 1 >$.

**Lemma 3.4 (Fox[1])** *For any integers $e_1$, $e_2$ and $e_3$ greater than 1, there are permutations $A$ and $B$ of order $e_1$ and $e_2$ respectively such that $AB$ has the order $e_3$.*

From this lemma, we take permutations $A$ and $B$ such as $e_1 = 2$, $e_2 = 3$ and $e_3 = e$ $(e \geq 6)$ and let $G$ be a finite group generatede by $A$ and $B$. Then there is a following exact sequence

$$1 \to Ker(\Phi) \to \pi_1(\mathbf{P}^1 - \{p_1, p_2, p_3\}, p_0) \xrightarrow{\Phi} G \to 1$$

So we have

$$\frac{\pi_1(\mathbf{P}^1 - \{p_1, p_2, p_3\}, p_0)}{Ker(\Phi)} \cong G.$$

Let $\ll \gamma_1^2, \gamma_2^3, \gamma_3^e \gg$ be the smallest normal subgroup of $\pi_1(\mathbf{P}^1 - \{p_1, p_2, p_3\}, p_0)$ which contains $\gamma_1^2, \gamma_2^3, \gamma_3^e$. Then we have $\ll \gamma_1^2, \gamma_2^3, \gamma_3^e \gg \subset Ker(\Phi)$ and there is a finite Galois covering $\pi : X_g \to \mathbf{P}^1$ which branches at $2p_1 + 3p_2 + ep_3$. Here the Galois group $Gal(\pi)$ of $\pi$ is naturally isomorphic to $G$ and we can deteremine the genus $g$ of the compact Riemann surface $X_g$ from the order $|G|$ of $G$ .

From Zariski-van Kampen theorem there is a following exact sequence

$$1 \to \ll \gamma_1^2, \gamma_2^3, \gamma_3^e \gg \to Ker(\Phi) \to \pi_1(X_g) \to 1.$$

So we have

$$\frac{Ker(\Phi)}{\ll \gamma_1^2, \gamma_2^3, \gamma_3^e \gg} \cong \pi_1(X_g).$$

It is easy to see that $\pi_1(\mathbf{P}^1 - \{p_1, p_2, p_3\}, p_0) / \ll \gamma_1^2, \gamma_2^3, \gamma_3^e \gg$ is isomorphic to $G(2,3,e)$. Combining above isomorphisms, we have the following exact sequence

$$1 \to \pi_1(X_g) \to G(2,3,e) \to G \to 1.$$

For any positive integer $q$, put

$$L(q) = \pi_1(X_g)^q[\pi_1(X_g), \pi_1(X_g)].$$

Then $L(q)$ is a characteristic normal subgroup of $\pi_1(X_g)$ of index $q^{2g}$ (cf. [2]). So $L(q)$ is a normal subgroup of $G(2,3,e)$ of index $|G| \times q^{2g}$. Note that $\cap_{q>0}L(q)$ is a subgroup of $G(2,3,e)$ of infinite index. We put $K(q)$ the pull-back of $L(q)$ to $\pi_1(\mathbf{P^2} - \{\bar{C} \cup L_\infty\})$. Then $K(q)$ is a normal subgroup of finite index of $\pi_1(\mathbf{P^2} - \{\overline{C} \cup L_\infty\})$ which contains $J_{e,m}$. Note that $\widetilde{K'} = \cap_{q>0}K(q)$ is of infinite index. In this case $C$ has only one singular point at the origin $p = (0,0)$ and the corresponding homomorphism $(i_p)_* : \pi_1(V_p - B, p'_0) \to \pi_1(M - B, p'_0)$ is an isomorphism. So $\widetilde{K}$ $(\subset \widetilde{K'})$ cannot satisfy the Condition2.3 .

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